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# *Heat Kernels*

by Wolfgang Arendt

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# Preface

Kernels for semigroups generated by elliptic operators play an important role for the study of parabolic equations. Most important are Gaussian estimates. They have striking consequences concerning spectral and regularity properties for the parabolic equations which are important for the study of nonlinear equations. Kernel estimates form an alternative approach which requires no or little regularity for the coefficients and the domain, in contrast to classical approaches where all data have to be $C^\infty$. In addition, there are elegant proofs with help of form methods. In the following 15 lectures these form methods will be presented and kernel estimates will be given with some applications. The aim is to show the typical ideas without heading for the most generality (which can be found in the excellent new monograph of Ouhabaz [Ouh05] and in the classical book by Davies [Dav89]).

The first lecture gives a brief introduction to semigroups with emphasis on some special features which are important in the sequel. The Spectral Theorem for selfadjoint operators will be used throughout to visualize the form methods. The following lectures will talk about the Laplacian for which, at least in the case of Dirichlet boundary conditions, very nice direct proofs can be given. The results are a guideline for the more general class of elliptic operators which will be treated by the form method.

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# Contents

# Lecture 1

# Unbounded Operators

In this lecture we introduce unbounded operators and put together some properties which will be frequently used. Moreover, we discuss the Spectral Theorem for selfadjoint operators which will give us very interesting examples of elliptic operators in the sequel.

## 1.1 Closed operators

Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

**Definition 1.1.1.** *An* **operator** *on $X$ is a linear mapping $A : D(A) \to X$, where $D(A)$ is a subspace of $X$ which we call the* **domain** *of $A$. The operator $A$ is called* **bounded** *if*

$$\|A\| := \sup_{\|x\| \le 1, x \in D(A)} \|Ax\| < \infty.$$

*If $\|A\| = \infty$, then $A$ is said to be* **unbounded**.

The notion of an operator is too general to allow one to do some analysis. The least thing one needs is to exchange limits and the operation. This is made precise in the following definition.

**Definition 1.1.2.** *An operator $A$ is* **closed** *if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ exist in $X$ one has $x \in D(A)$ and $Ax = y$.*

Thus an operator $A$ on $X$ is closed if and only if its **graph**

$$G(A) := \{(x, Ax) : x \in D(A)\}$$

is a closed subspace of $X \times X$.

If $D(A)$ is a closed subspace of $X$, then the closed graph theorem asserts that $A$ is bounded if and only if $A$ is closed. We will be mainly interested in closed operators with dense domain.

In order to give a first typical example we need the following.

**Remark 1.1.3** (graph norm)**.** Let $A$ be an operator on $X$. Then

$$\|x\|_A = \|x\| + \|Ax\|$$

defines a norm on $D(A)$ which we call the **graph norm**. The operator $A$ is closed if and only if $(D(A), \|\cdot\|_A)$ is a Banach space.

**Definition 1.1.4** (extension of operators)**.** *Let $A, B$ be two operators on $X$.*

1. *We say that $B$ is an* **extension** *of $A$ and write $A \subset B$ if*

$$\begin{aligned} D(A) &\subset D(B) \quad \text{and} \\ Ax &= Bx \quad \text{for all } x \in D(A). \end{aligned}$$

2. *Two operators $A$ and $B$ are said to be* **equal** *if $A \subset B$ and $B \subset A$, i.e., if $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$.*

## 1.2 The spectrum

Let $X$ be a complex Banach space. Let $A$ be an operator on $X$. Frequently, even if $A$ is unbounded, it might have a bounded inverse. In that case, we may use properties and theorems on bounded operators to study $A$.

For this, frequently it does not matter if $A$ is replaced by $\lambda I - A$ where $\lambda \in \mathbb{C}$ and $I$ is the identity operator on $X$. The set

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is bijective and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

is called the **resolvent set** of $A$. Here $\mathcal{L}(X)$ is the space of all bounded operators from $X$ into $X$. If $\lambda I - A : D(A) \to X$ is bijective, then $(\lambda I - A)^{-1} : X \to D(A)$ is linear. But in the definition we ask in addition that $(\lambda I - A)^{-1}$ is a bounded operator from $X$ into $X$. This is automatic if $A$ is closed.

**Proposition 1.2.1** (closed operators and resolvents)**.** *Let $A$ be an operator on $X$.*

1. *Let $\lambda \in \mathbb{C}$. Then $A$ is closed if and only if $(\lambda - A)$ is closed.*

2. *If $\rho(A) \neq \emptyset$, then $A$ is closed.*

3. *Assume that $A$ is closed and $(\lambda - A) : D(A) \to X$ is bijective. Then $\lambda \in \rho(A)$.*

We omit the easy proof.

For $\lambda \in \rho(A)$, the operator

$$R(\lambda, A) = (\lambda I - A)^{-1} \in \mathcal{L}(X)$$

is called the **resolvent** of $A$ in $\lambda$.

We frequently write $(\lambda - A)$ as shorthand for $(\lambda I - A)$. The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the **spectrum** of $A$.

If $B \in \mathcal{L}(X)$, then $\rho(B) \neq \emptyset$. In fact, assume that $\|B\| < 1$. Then

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \qquad (\textbf{Neumann series}). \tag{1.1}$$

Replacing $B$ by $\frac{1}{\lambda}B$ one sees that $\lambda \in \rho(B)$ whenever $|\lambda| > \|B\|$.

Unbounded closed operators may have empty resolvent set. Also, it may happen that an unbounded operator has empty spectrum (which is not true for operators in $\mathcal{L}(X)$).

**Proposition 1.2.2** (analyticity of the resolvent)**.** *Let $A$ be an operator and $\lambda_0 \in \rho(A)$. If $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}$, then $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$$

*which converges in $\mathcal{L}(X)$. Consequently,* $\mathrm{dist}(\lambda_0, \sigma(A)) \geq \|R(\lambda_0, A)\|^{-1}$.

*Proof.* One has $(\lambda - A) = (\lambda - \lambda_0) + (\lambda_0 - A) = (I - (\lambda_0 - \lambda)R(\lambda_0, A))(\lambda_0 - A)$. Since $\|(\lambda_0 - \lambda)R(\lambda_0, A)\| < 1$, the operator $((I - (\lambda_0 - \lambda)R(\lambda_0, A))$ is invertible and its inverse is given by

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^n.$$

Hence $\lambda \in \rho(A)$ and

$$R(\lambda, A) = R(\lambda_0, A)(I - (\lambda_0 - \lambda)R(\lambda_0, A))^{-1}.$$

This concludes the proof. □

By

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \ \exists x \in D(A), x \neq 0, (\lambda - A)x = 0\}$$

we denote the **point spectrum**, or the set of all **eigenvalues** of $A$. If $\lambda$ is an eigenvalue, each $x \in D(A) \setminus \{0\}$ such that $(\lambda - A)x = 0$ is called an **eigenvector** of $A$. There is a natural relation between the spectrum of $A$ and its resolvents.

**Proposition 1.2.3** (Spectral Mapping Theorem for resolvents)**.** *Let $\lambda_0 \in \rho(A)$. Then*

1. $\sigma(R(\lambda_0, A)) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(A)\}$,
2. $\sigma_p(R(\lambda_0, A)) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma_p(A)\}$.

*Proof.* a) 1. If $\mu \in \rho(A), \mu \neq \lambda_0$, then

$$\left(\frac{1}{\lambda_0 - \mu} - R(\lambda_0, A)\right)^{-1} = (\lambda_0 - \mu)(\lambda_0 - A)R(\mu, A).$$

2. " $\subset$ " Let $\nu \in \sigma(R(\lambda_0, A)), \nu \neq 0$. Assume that $\nu \notin \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(A)\}$. Then $\lambda_0 - 1/\nu \in \rho(A)$. This implies $\nu \in \rho(R(\lambda_0, A))$ by 1.

3. " $\supset$ " Let $\mu = (\lambda_0 - \lambda)^{-1}$ where $\lambda \neq \lambda_0$. Suppose that $\mu \in \rho(R(\lambda_0, A))$. Then one easily sees that $\lambda \in \rho(A)$ and $R(\lambda, A) = \mu R(\lambda_0, A)(\mu - R(\lambda_0, A))^{-1}$.

b) is left to the reader. □

We conclude this section by the following crucial identity.

**Proposition 1.2.4** (Resolvent Identity)**.**

$$(R(\lambda, A) - R(\mu, A))/(\mu - \lambda) = R(\lambda, A)R(\mu, A) \tag{1.2}$$

*for all* $\lambda, \mu \in \rho(A), \lambda \neq \mu$.

*Proof.* One has

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= R(\lambda, A)[I - (\lambda - A)R(\mu, A)] \\ &= R(\lambda, A)[(\mu - A) - (\lambda - A)]R(\mu, A) \\ &= (\mu - \lambda)R(\lambda, A)R(\mu, A). \end{aligned}$$

□

The resolvent identity shows in particular that resolvents commute.

It is convention to define the spectrum always with respect to complex spaces. There are good reasons for this. For instance, the well-known fact that bounded operators on a complex Banach space have non-empty spectrum is no longer valid for real spaces (even in dimension 2). More important, for our purposes, spectral theory allows one to deduce asymptotic properties of orbits from spectral properties of the operators. For this it is important that the underlying field is $\mathbb{C}$.

If an operator $A$ is given on a real Banach space $X$, then it has a unique $\mathbb{C}$-linear extension $A_{\mathbb{C}}$ to the complexification $X_{\mathbb{C}} = X \oplus iX$ of $X$. Then $X_{\mathbb{C}}$ is a Banach space for the norm

$$\|z\|_{E_{\mathbb{C}}} := \sup_{|\lambda| \leq 1} \|\mathrm{Re}(\lambda z)\|,$$

where $z = x + iy \in X_{\mathbb{C}}$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$. In many cases, there is a more natural equivalent norm. For example, if $X$ is the real space $L^p(\Omega)$, where $(\Omega, \Sigma, \mu)$ is a measure space, then $X_{\mathbb{C}}$ is the complex space $L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega,\mathbb{C})} = (\int_\Omega |f|^p d\mu)^{\frac{1}{p}}$$

is the natural norm. Now the domain $D(A_{\mathbb{C}}) = D(A) \oplus iD(A)$ is the complexfication of the domain. Then $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \to X_{\mathbb{C}}$ is defined by $A_{\mathbb{C}}(x+iy) := Ax + iAy$ where $x, y \in D(A)$.

**Definition 1.2.5.** *Let $A$ be an operator on a real Banach space $X$. Then we define $\sigma(A) := \sigma(A_{\mathbb{C}}), \rho(A) := \rho(A_{\mathbb{C}})$.*

Note that this definition is consistent in the following two points. If $\lambda \in \rho(A) \cap \mathbb{R}$, then $R(\lambda, A)$ leaves invariant the real space $X$ and $R(\lambda, A)_{|X}$ is the inverse for $\lambda - A$ on $X$. If $\lambda \in \rho(A) \cap \mathbb{R}$ we will frequently use the notation $R(\lambda, A) = (\lambda - A)^{-1}$ for the operator restricted to $X$ instead of writing $R(\lambda, A)_{|X}$. Moreover,

$$\sigma_p(A) \cap \mathbb{R} = \{\lambda \in \mathbb{R} : \exists u \in D(A), u \neq 0, Au = \lambda u\},$$

i.e., $\sigma_p(A) \cap \mathbb{R}$ consists of the eigenvalues of the operator $A$ on $X$.

## 1.3 Operators with compact resolvent

Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By $\mathcal{K}(X)$ we denote the space of all compact operators on $X$. The following facts are well-known.
$\mathcal{K}(X)$ is a closed subspace of $\mathcal{L}(X)$. It is even an ideal, i.e., $K \in \mathcal{K}(X)$ implies $SK,\ KS \in \mathcal{K}(X)$ for all $S \in \mathcal{L}(X)$.

Compact operators have very particular spectral properties. Let $K \in \mathcal{K}(X)$. Then the spectrum consists only of eigenvalues with 0 as possible exception, i.e.,

$$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}\ . \tag{1.3}$$

Moreover, $\sigma(K)$ is countable with 0 as only possible accumulation point, i.e., either $\sigma(K)$ is finite or there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$ and

$$\sigma(K) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}.$$

Finally, for each $\lambda \in \sigma_p(K) \setminus \{0\}$, the eigenspace $\ker(\lambda - K)$ is finite dimensional.

The purpose of this section is to find out what all these properties mean for an unbounded operator if its resolvent is compact.

**Definition 1.3.1.** *An operator $A$ on $X$ has* **compact resolvent** *if $\rho(A) \neq \emptyset$ and $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$.*

From the resolvent identity and the ideal property, it follows that $A$ has compact resolvent whenever $R(\lambda, A) \in \mathcal{K}(X)$ for some $\lambda \in \rho(A)$.

If $\dim X = \infty$ then operators with compact resolvent are necessarily unbounded (otherwise, for $\lambda \in \rho(A)$ we have $R(\lambda, A) \in \mathcal{K}(X)$ and $(\lambda - A) \in \mathcal{L}(X)$. Thus $I = (\lambda - A)R(\lambda, A)$ is compact by the ideal property).

The following spectral properties follow easily from those of compact operators with help of the Spectral Mapping Theorem for resolvents (Propositon 1.2.3).

**Proposition 1.3.2** (spectral properties of operators with compact resolvent). *Let $A$ be an operator with compact resolvent. Then*

1. $\sigma(A) = \sigma_p(A)$;
2. *either $\sigma(A)$ is finite or there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $\lim_{n\to\infty} |\lambda_n| = \infty$ and $\sigma(A) = \{\lambda_n : n \in \mathbb{N}\}$;*
3. $\dim\ker(\lambda - A) < \infty$ *for all $\lambda \in \mathbb{C}$ where* $\ker(\lambda - A) := \{x \in D(A) : Ax = \lambda x\}$.

The most simple examples of unbounded operators are diagonal operators. In the next section we will see that selfadjoint operators with compact resolvent are equivalent to such simple operators. We let

$$\ell^2 := \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{K} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

where as usual $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\ell^2$ is a separable Hilbert space over $\mathbb{K}$ with respect to the scalar product

$$(x \mid y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

(where $\bar{y}_n$ is the complex conjugate of $y_n$).

**Definition 1.3.3** (diagonal operator). *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$. The operator $M_\alpha$ on $\ell^2$ given by*

$$\begin{aligned} D(M_\alpha) &= \{x \in \ell^2 : (\alpha_n x_n)_{n\in\mathbb{N}} \in \ell^2\} \\ M_\alpha x &= (\alpha_n x_n)_{n\in\mathbb{N}} \end{aligned}$$

*is called the* **diagonal operator** *associated with $\alpha$ and is denoted by $M_\alpha$.*

We define the sequence spaces

$$\begin{aligned} \ell^\infty &= \{\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{K} : \sup_{n\in\mathbb{N}} |\alpha_n| < \infty\}, \\ c_0 &= \{\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{K} : \lim_{n\to\infty} |\alpha_n| = 0\}. \end{aligned}$$

Note that $\ell^\infty$ is a Banach space for the norm

$$\|\alpha\|_\infty = \sup_{n\in\mathbb{N}} |\alpha_n|$$

and $c_0$ is a closed subspace.

**Proposition 1.3.4** (properties of diagonal operators)**.** *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. *Then*

1. *the operator* $M_\alpha$ *is closed;*
2. *the operator* $M_\alpha$ *is bounded if and only if* $\alpha \in \ell^\infty$*;*
3. $\sigma(M_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}}$*;*
4. *the operator* $M_\alpha$ *is compact if and only if* $\alpha \in c_0$*;*
5. *the operator* $M_\alpha$ *has compact resolvent if and only if* $\lim_{n\to\infty} |\alpha_n| = \infty$.

We leave the proof as exercise.

Given an operator $A$, it is easy to define by similarity a new operator that has the same properties as $A$.

**Proposition 1.3.5** (similarity)**.** *Let $A$ be an operator on $X$ and let $V : X \to Y$ be an isomorphism where $Y$ is a Banach space. Define the operator $B$ on $Y$ by*

$$\begin{aligned} D(B) &= \{y \in Y : V^{-1}y \in D(A)\} \\ By &= VAV^{-1}y. \end{aligned}$$

*Then*

1. *$A$ is closed if and only if $B$ is closed;*
2. $\rho(B) = \rho(A)$ *and* $R(\lambda, B) = VR(\lambda, A)V^{-1}$ *for all* $\lambda \in \rho(B)$*;*
3. *$B$ has compact resolvent if and only if $A$ has compact resolvent.*

**Notation**: $VAV^{-1} := B$. The easy proof is left as exercise.

## 1.4 Selfadjoint operators with compact resolvent

Here we consider unbounded operators on a Hilbert space. The main result is the Spectral Theorem which shows that every selfadjoint operator with compact resolvent can be represented as a diagonal operator.
We treat the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ simultaneously. Whereas some spectral theoretical theorems need complex Banach spaces, the Spectral Theorem below is true on real and on complex Hilbert space. Throughout this section $H$ is an infinite dimensional Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

**Definition 1.4.1.** *An operator $A$ on $H$ is called* **dissipative** *if*

$$\mathrm{Re}(Ax|x) \leq 0 \textit{ for all } x \in D(A).$$

The following proposition shows a remarkable spectral property of dissipative operators.

**Proposition 1.4.2.** *Let $A$ be a dissipative operator on $H$. Assume that there exists $\lambda \in \mathbb{K}, \operatorname{Re}\lambda > 0$ such that $(\lambda - A)$ is surjective. Then $\mu \in \rho(A)$ and $\|R(\mu, A)\| \le 1/\operatorname{Re}\mu$ for all $\mu \in \mathbb{K}$ such that $\operatorname{Re}\mu > 0$.*

*Proof.* Let $\mu \in M := \{\mu \in \rho(A) : \operatorname{Re}\mu > 0\}$. Let $x \in D(A)$, $\mu x - Ax = y$. Then

$$\begin{array}{rcccl} \mathrm{Re}\mu\|x\|^2 & = & \mathrm{Re}(\mu x|x) & = & \mathrm{Re}(y + Ax|x) \\ & = & \mathrm{Re}(x|y) & + & \mathrm{Re}(Ax|x) \\ & \le & \mathrm{Re}(x|y) & \le & \|x\|\,\|y\| \end{array}$$

by dissipativity and the Cauchy–Schwartz inequality. Thus $(\mathrm{Re}\mu)\|x\| \le \|y\|$. It follows that

$$\|R(\mu, A)\| \le \frac{1}{\mathrm{Re}\mu} \tag{1.4}$$

whenever $\mu \in M$. Since $\rho(A)$ is open, also $M$ is open. Observe now that, as a consequence of Proposition 1.2.2, if $\lambda_n \in \rho(A)$ and $\lambda = \lim_{n\to\infty} \lambda_n$ such that

$$\sup_{n\in\mathbb{N}} \|R(\lambda_n, A)\| < \infty,$$

then $\lambda \in \rho(A)$. This and (1.4) imply that $M$ is closed in the right open half-plane (half-line if $\mathbb{K} = \mathbb{R}$), which is connected. Since the right half-plane $M$ is connected and non-empty, it follows that $M$ is the entire open half-plane (the open right half-line if $\mathbb{K} = \mathbb{R}$). ☐

**Definition 1.4.3.** *An operator $A$ on $H$ is called* **m-dissipative** *if $A$ is dissipative and $(I-A)$ is surjective.*

From Proposition 1.4.2 we know that the spectrum of an $m$-dissipative operator $A$ is contained in the left half-plane, and $\|R(\lambda, A)\| \le 1/\mathrm{Re}\lambda$ $(\mathrm{Re}\lambda > 0)$.

Now we consider the more special class of $m$-dissipative symmetric operators.

**Definition 1.4.4.** *An operator $A$ on $H$ is called* **symmetric** *if*

$$(Ax|y) = (x|Ay) \text{ for all } x, y \in D(A).$$

If $A$ is bounded, then $A$ is symmetric if and only if $A = A^*$, i.e., if $A$ is **selfadjoint**. The class of $m$-dissipative symmetric operators is important in the sequel. These are exactly the generators of contractive $C_0$-semigroups of selfadjoint operators as we will see later. Because of their importance we recollect the properties in the following definition.

**Definition 1.4.5.** *An operator $A$ on $H$ is $m$-dissipative and symmetric if*

1. $(Ax \mid y) = (x \mid Ay) \qquad (x, y \in D(A)) \qquad$ (**symmetry**),
2. $(Ax \mid x) \leq 0 \qquad (x \in D(A)) \qquad$ (**form-negativity**),
3. $\forall y \in H\ \exists\, x \in D(A)$ *such that* $x - Ax = y \qquad$ (**range condition**).

An $m$-dissipative, symmetric operator is also called a **dissipative selfadjoint operator**. Observe that $(Ax \mid x) \in \mathbb{R}$ by a). It follows from Proposition 1.4.2 that $(0, \infty) \subset \rho(A)$ and that $R(\lambda, A)$ is symmetric for all $\lambda > 0$ whenever $A$ is dissipative and selfadjoint. We will apply the Spectral Theorem for bounded selfadjoint operators to $R(\lambda, A)$ in order to characterize these operators as diagonal or as multiplication operators.

**Example 1.4.6** ($m$-dissipative symmetric diagonal operators)**.** *Let $\lambda := (\lambda_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lambda_n \leq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = -\infty$. Then the diagonal operator $M_\lambda$ on $\ell^2$ given by*

$$\begin{aligned} M_\lambda x &:= (\lambda_n x_n)_{n\in\mathbb{N}} \\ D(M_\lambda) &:= \{x \in \ell^2 : (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\} \end{aligned}$$

*is $m$-dissipative, symmetric and has compact resolvent.*

We recall the Spectral Theorem for compact symmetric operators.

**Proposition 1.4.7.** *Let $B$ be a compact, symmetric operator on a separable Hilbert space $H$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $H$ has an orthonormal basis which consists of eigenvectors of $B$.*

This result is contained in all standard texts on Functional Analysis (see e.g. [RS72, p. 203]).

For our purpose the following version for unbounded operators is important.

**Theorem 1.4.8** (Spectral Theorem: diagonal form)**.** *Let $A$ be an $m$-dissipative, symmetric operator with compact resolvent on a separable Hilbert space $H$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Assume that $\dim H = \infty$. Then there exist an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $H, \lambda_n \in \mathbb{R}, \lambda_n \leq 0$, such that $e_n \in D(A)$, $Ae_n = \lambda_n e_n$ and $\lim_{n\to\infty} \lambda_n = -\infty$. Moreover, $A$ is given by*

$$\begin{aligned} D(A) &= \{x \in H : (\lambda_n (x|e_n))_{n\in\mathbb{N}} \in \ell^2\} \\ Ax &= \sum_{n=1}^{\infty} \lambda_n (x|e_n) e_n. \end{aligned}$$

*Proof.* Since $A$ has compact resolvent, by Proposition 1.3.2 there exists $\mu \in (0,\infty)\cap\rho(A)$. Then $R(\mu,A)$ is a compact and symmetric operator (as is easy to see). By Proposition 1.4.7 there exists an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $H$ and $\alpha_n \in \mathbb{R}$ such that $R(\mu,A)e_n = \alpha_n e_n$. Since $R(\mu,A)$ is injective one has $\alpha_n \neq 0$ $(n \in \mathbb{N})$. Hence $e_n \in D(A)$ and $e_n = \alpha_n(\mu - A)e_n$. It follows that $Ae_n = \lambda_n e_n$ where $\lambda_n = (\mu - \frac{1}{\alpha_n})$. Moreover, $\lambda_n = (Ae_n \mid e_n) \leq 0$ since $A$ is dissipative. Since $\lim\limits_{n\to\infty} |\alpha_n| = 0$, one has $\lim\limits_{n\to\infty} \lambda_n = -\infty$. Let $x \in D(A)$. Then $(\lambda_n(x|e_n))_{n\in\mathbb{N}} = ((x|Ae_n))_{n\in\mathbb{N}} \in \ell^2$ and $Ax = \sum\limits_{n=1}^{\infty}(Ax|e_n)e_n = \sum\limits_{n=1}^{\infty} \lambda_n(x|e_n)e_n$. Conversely, assume that $x \in H$ such that $(\lambda_n(x|e_n))_{n\in\mathbb{N}} \in \ell^2$. Let $x_m = \sum\limits_{n=1}^{m}(x|e_n)e_n, y_m = \sum\limits_{n=1}^{m} \lambda_n(x|e_n)e_n$. Then $\lim_{m\to\infty} x_m = x$ and $y_m$ converges as $m \to \infty$. Observe that $x_m \in D(A)$ and $Ax_m = y_m$. Since $A$ is closed, it follows that $x \in D(A)$. $\square$

There is another way to present the Spectral Theorem. Denote by $U : H \to \ell^2$ the unitary operator given by $Ux = ((x|e_n))_{n\in\mathbb{N}}$. Then it follows directly from Theorem 1.4.8 that

$$UAU^{-1} = M_\lambda \tag{1.5}$$

(see Proposition 1.6 for the notation). We have obtained the following result.

**Corollary 1.4.9** (diagonalization)**.** *Let $A$ be an operator on $H$. Suppose that $\dim H = \infty$. The following assertions are equivalent.*

*(i) $A$ is m-dissipative, symmetric and has compact resolvent;*

*(ii) there exists a unitary operator $U : H \to \ell^2$ and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_n \leq 0$, $\lim\limits_{n\to\infty} \lambda_n = -\infty$ and*

$$UAU^{-1} = M_\lambda\ .$$

We express (ii) by saying that $A$ and $M_\lambda$ are **unitarily equivalent**.

The Spectral Theorem establishes a surprising metamorphoses. Frequently the operator $A$ will be given as a differential operator. But identifying $H$ with $\ell^2$ via the unitary operator $U$, the operator $A$ is transformed into the diagonal operator $M_\lambda$. This will be most convenient to prove properties of $A$.

**Remark 1.4.10** (complexification of real Hilbert spaces)**.** *Let $H$ be a separable real Hilbert space. On $H_\mathbb{C}$ we consider the scalar product*

$$(x_1 + iy_1 \mid x_2 + iy_2)_{H_\mathbb{C}} = (x_1 \mid x_2) + (y_1 \mid y_2) + i(y_1 \mid x_2) - i(x_1 \mid y_2).$$

*Then the following assertions hold.*

1. *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis which allows us to identify $H$ and the real space $\ell^2$. Then $H_{\mathbb{C}}$ can be identified with the complex space $\ell^2$.*

2. *Let $A$ be a dissipative operator on $H$. Then $A_{\mathbb{C}}$ is dissipative.*

## 1.5 The Spectral Theorem

In this section we give a representation of $m$-dissipative symmetric operators which do not necessarily have a compact resolvent. Indeed, a simple example of a selfadjoint operator is obtained if we consider multiplication by a function in $L^2$ instead of a sequence in $\ell^2$. We make this more precise.

**Proposition 1.5.1** (multiplication operators)**.** *Let $(Y, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $m : Y \to (-\infty, 0]$ be a measurable function. Define the operator $A_m$ on $L^2(Y, \Sigma, \mu)$ by*

$$\begin{aligned} D(A_m) &= \{f \in L^2(Y, \Sigma, \mu) : mf \in L^2(Y, \Sigma, \mu)\} \\ A_m f &= mf \ . \end{aligned}$$

*Then $A_m$ is $m$-dissipative and symmetric.*

This is not difficult to see. Here $L^2(Y, \Sigma, \mu)$ may be the real or complex space.

Of course multiplication operators contain diagonal operators as special case: It suffices to take $Y = \mathbb{N}$ and $\mu$ the counting measure. But they are more general. In fact, each diagonal operator has eigenvalues whereas a multiplication operator does not, in general (see Exercise 1.7.4). And indeed, multiplication operators are the most general selfadjoint operators as the following theorem shows (see also Theorem 1.8.4 in the comments).

**Theorem 1.5.2** (Spectral Theorem: multiplication form)**.** *Let $A$ be an $m$-dissipative symmetric operator on a separable Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist a finite measure space $(Y, \Sigma, \mu)$, a measurable function $m : Y \to \mathbb{R}$ and a unitary operator $U : H \to L^2(Y, \Sigma, \mu)$ such that*

$$UAU^{-1} = A_m \ . \tag{1.6}$$

We refer to Proposition 1.3.5 for the notation used in (1.6). For the proof of Theorem 1.5.2 we refer to [RS72, Theorem VIII.4, p. 260].

## 1.6 Hilbert–Schmidt operators

Here we discuss briefly a classical class of kernel operators. Let $\Omega \subset \mathbb{R}^n$ be an open set and $H = L^2(\Omega)$ with respect to the Lebesgue measure. Throughout this section the underlying field may be $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

**Theorem 1.6.1.** *Let $T$ be a linear operator on $H$. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $H$. Then the following are equivalent:*

1. *there exists $k \in L^2(\Omega \times \Omega)$ such that*
$$Tf(x) = \int_\Omega k(x,y)f(y)dy \quad \textit{for a.e. } x \in \Omega,$$
*for all $f \in L^2(\Omega)$;*

2. $\sum_{n=1}^\infty \|Te_n\|_H^2 < \infty$.

*In that case $T$ is compact and one has*
$$\|T\|_{\mathcal{L}(H)} \le \|k\|^2_{L^2(\Omega\times\Omega)} = \sum_{n=1}^\infty \|Te_n\|_H^2.$$

We refer to [RS72, Thm. VI.6] for the proof.

The theorem shows in particular that condition b) is independent of the choice of the orthonormal basis. An operator $T$ on $H$ is called **Hilbert–Schmidt** if it satisfies the equivalent conditions of Theorem 1.6.1. The function $k \in L^2(\Omega \times \Omega)$ of condition b) is uniquely determined by $T$: we call it the **kernel** of $T$. The adjoint $T^*$ of a Hilbert–Schmidt operator is Hilbert–Schmidt again and its kernel $k^*$ is given by $k^*(x,y) = \overline{k(y,x)}$. In particular, $T$ is self-adjoint if and only if $k(x,y) = \overline{k(y,x)}$ a.e. on $\Omega \times \Omega$.

If $T$ is a compact selfadjoint operator, then $H$ has an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of eigenvectors.Then $T$ is Hilbert–Schmidt if and only if $\sum_{n=1}^\infty |\lambda_n|^2 < \infty$, where $Te_n = \lambda_n e_n$.

Our main point in this section is the following useful criterion.

**Theorem 1.6.2.** *Assume $\Omega$ to have finite Lebesgue measure. Let $T \in \mathcal{L}\big(L^2(\Omega)\big)$ such that*
$$TL^2(\Omega) \subset L^\infty(\Omega).$$
*Then $T$ is a Hilbert–Schmidt operator.*

For the proof we need the following special case of the Gelfand–Naimark Theorem (see [RS72, Thm. VII.1] or [Rud91, Thm. VII.1]). See the comments for another more direct proof.

**Theorem 1.6.3.** *There exists a compact space $K$ and a bijective linear mapping $\Phi : L^\infty(\Omega) \to C(K)$ such that*

1. $\Phi(f \cdot g) = \Phi(f)\Phi(g)$,

2. $\Phi(1_\Omega) = 1_K$,

*3.* $\Phi(\overline{f}) = \overline{\Phi(f)}$, *and*

*4.* $f \geq 0$ *a.e. if and only if* $\Phi(f) \geq 0$.

*5.* $\|\Phi(f)\|_{C(K)} = \|f\|_{L^\infty(\Omega)}$ *for all* $f \in L^\infty(\Omega)$.

*Proof of Theorem 1.6.2.* We consider $T$ as a linear operator from $L^2(\Omega)$ in $L^\infty(\Omega)$. It follows from the Closed Graph Theorem that $T$ is continuous. By Riesz's Theorem we find a finite Borel measure $\nu$ on $K$ such that

$$\int_K g(y)d\nu(y) = \int_\Omega (\Phi^{-1}g)(x)dx$$

for all $g \in C(K)$. Let

$$\widetilde{T} := \Phi \circ T : L^2(\Omega, \mu) \to C(K).$$

For $y \in K$ denote by $\delta_y$ the Dirac measure. Then $\widetilde{T}'\delta_y := \delta_y \circ \widetilde{T} \in L^2(\Omega)$. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\Omega)$. If $f \in L^2(\Omega)$, then by Parseval's identity

$$\|f\|^2_{L^2(\Omega)} = \|\overline{f}\|^2_{L^2(\Omega)} = \sum_{n=1}^\infty \left| \int_\Omega e_n(x) f(x) dx \right|^2.$$

Using this for $f = \widetilde{T}'\delta_y$ one obtains

$$\begin{aligned}
\sum_{n=1}^\infty \|Te_n\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty \int_K \left|(\widetilde{T}e_n)(y)\right|^2 d\nu(y) \\
&= \int_K \sum_{n=1}^\infty \left| \int_\Omega e_n(x)(\widetilde{T}'\delta_y)(x)dx \right|^2 d\nu(y) \\
&= \int_K \|\widetilde{T}'\delta_y\|^2_{L^2(\Omega)} d\nu(y) \\
&\leq \|\widetilde{T}'\|^2 \nu(K).
\end{aligned}$$

This concludes the proof. □

## 1.7 Exercises

**Exercise 1.7.1** (criterion for compact resolvent)**.** *Let $A$ be an operator on a Banach space $X$ with non-empty resolvent set. Then $A$ has compact resolvent if and only if the canonical injection $(D(A), \|\cdot\|_A) \hookrightarrow X$ is compact.*

Notice: if $\mathbb{K} = \mathbb{R}$, then $D(A) \hookrightarrow X$ is compact if and only if $D(A_\mathbb{C}) \hookrightarrow X_\mathbb{C}$ is compact.

**Exercise 1.7.2** (automatic density of the domain)**.** *Let $A$ be an $m$-dissipative operator on a Hilbert space $H$. Show that $D(A)$ is dense.*

*Hint:* By Hilbert space theory it suffices to show that $D(A)^{\perp} := \{y \in H : (x|y) = 0$ for all $x \in D(A)\} = 0$.

**Exercise 1.7.3** (dissipativity again)**.** *Let $A$ be an operator on a Hilbert space $H$. Show that $A$ is dissipative if and only if*

$$\|x - tAx\| \geq \|x\| \tag{1.7}$$

*for all $x \in D(A), t > 0$.*

**Exercise 1.7.4** (no eigenvalues)**.** *Let $Y = \mathbb{R}$ with Lebesgue measure, $m(y) = y \quad (y \in \mathbb{R}),\ H = L^2(\mathbb{R})$. Then $\sigma_p(A_m) = \emptyset$. Deduce from this that $A_m$ is not unitarily equivalent to a diagonal operator.*

Let $A$ be a densely defined operator on a Hilbert space $H$. We define the adjoint $A^*$ of $A$ by

$$\begin{aligned} D(A^*) &:= \{x \in H : \exists z \in H \text{ such that } (Au \mid x)_H = (u \mid z)_H\ \forall u \in D(A)\} \\ A^*x &:= z. \end{aligned}$$

**Exercise 1.7.5** (adjoint of $m$-dissipative operator)**.** *Let $A$ be an operator on a Hilbert space $H$. Show the following.*

1. *$A$ is $m$-dissipative if and only if $(0, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1 \qquad (\lambda > 0)$.*
2. *If $A$ is $m$-dissipative, then $A^*$ is $m$-dissipative and $R(\lambda, A)^* = R(\lambda, A^*)$.*
3. *If $A$ is $m$-dissipative and symmetric, then $A = A^*$.*

**Exercise 1.7.6** (spectrum and essential image)**.** *Let $A_m$ be a multiplication operator on $L^2(Y, \Sigma, \mu)$ (see Proposition 1.5.1). Show that*

$$\sigma(A_m) = \text{ ess image } (m)$$

*where the* **essential image** *of $m$ is defined by*

$$\text{ess image } (m) := \{\lambda \in \mathbb{C} : \forall\ \varepsilon > 0\ \mu(\{x : |m(x) - \lambda| \leq \varepsilon\}) > 0\}.$$

# 1.8 Comments

In these comments we give some additional information on the material of each lecture. They are not needed in the sequel but frequently illuminating. Here we will explain in particular how the the special class of $m$-dissipative symmetric operators is related to the general class of all selfadjoint operators.

## 1.8.1 $m$-dissipative operators

Let $A$ be a dissipative operator on a Hilbert space. Then $A$ is called **maximal dissipative** if the following holds: If $A \subset B$ where $B$ is a dissipative operator, then $A = B$. It turns out that $m$-dissipativity (i.e., dissipativity and range condition) is the same as maximal dissipativity. Moreover, each dissipative operator has an $m$-dissipative extension. These are properties which merely hold on Hilbert spaces and not on Banach spaces (where dissipativity may be defined by (1.7)).

### 1.8.2 General selfadjoint operators

In this comment we define general selfadjoint operators which are not necessarily dissipative. Let $H$ be a complex Hilbert space and $A$ an operator on $H$. If $A$ is symmetric, then $(Ax|x) = (x|Ax) = \overline{(Ax|x)}$. Hence $(Ax|x) \in \mathbb{R}$ for all $x \in D(A)$. Also the converse is true. This follows from the **Polarisation Identity**.

$$(x|y) = \tfrac{1}{4}\{(x+y, x+y) - (x-y|x-y) + i(x+iy|x+iy) - i(x-iy|x-iy)\} \tag{1.8}$$

$(x, y \in H)$, which is an immediate consequence of the properties of the scalar product. In fact, considering $y = Ax$ in (1.8) one sees the following.

**Proposition 1.8.1.** *Let $A$ be an operator on a complex Hilbert space $H$. The following assertions are equivalent:*

*(i)* *$A$ is symmetric;*

*(ii)* *$(Ax|x) \in \mathbb{R}$ for all $x \in D(A)$;*

*(iii)* *$\pm iA$ is dissipative.*

Note that (iii) is just a reformulation of (ii). But now Proposition 1.4.2 applied to $iA$ shows us the following.

**Proposition 1.8.2.** *Let $A$ be a symmetric operator on a complex Hilbert space. Assume that $(\lambda - A)$ is surjective for some $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda > 0$. Then $\lambda \in \rho(A)$ for all $\lambda$ with $\mathrm{Im}\lambda > 0$. Similarly, if $(\lambda - A)$ is surjective for some $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda < 0$, then $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda < 0\} \subset \rho(A)$.*

Thus for a symmetric operator $A$, there are four possibilities:

1. $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \geq 0\}$,
2. $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \leq 0\}$,
3. $\sigma(A) = \mathbb{C}$, or
4. $\sigma(A) \subset \mathbb{R}$.

The cases (a)–(c) are not of interest for our purposes and we refer to the literature for further investigation (e.g. [RS72]). We are rather interested in the last case (d) which leads to the following definition.

**Definition 1.8.3.** *An operator $A$ on a complex Hilbert space is called* **selfadjoint** *if $A$ is symmetric and if $(i - A)$ and $(-i - A)$ are surjective.*

By our discussion, a selfadjoint operator has real spectrum. Whereas every bounded symmetric operator is selfadjoint, for unbounded operators, this is not true, and the range condition ($(\pm i - A)$ to be surjective) is a severe restriction.
An operator $A$ is symmetric if and only if $A \subset A^*$ and $A$ is selfadjoint if and only if $A = A^*$.
The Spectral Theorem also holds without the assumption of dissipativity which we made in the lecture having in mind the application to evolution equations. The general theorem looks as follows.

**Theorem 1.8.4** (Spectral Theorem (general form)). *Let $A$ be a selfadjoint operator on a separable Hilbert space $H$. Then there exist a finite measure space $(Y, \Sigma, \mu)$, a unitary operator $U : H \to L^2(Y, \Sigma, \mu)$ and a measurable function $m : Y \to \mathbb{R}$ such that*

$$UAU^{-1} = A_m$$

*where*

$$\begin{aligned} A_m f &= mf \\ D(A_m) &= \{f \in L^2(Y, \Sigma, \mu) : mf \in L^2(X, \Sigma, \mu)\}. \end{aligned}$$

*Conversely, every $A_m$ and so every operator $A = U^{-1} A_m U$ is selfadjoint.*

The operator $A$ is dissipative if and only if $\sigma(A) \subset (-\infty, 0]$. In fact, $\sigma(A) = \sigma(A_m) =$ ess image $(m)$ by Exercise 1.7.6.

## 1.8.3 Quantum Theory

In Quantum Theory an observable is modelised by a selfadjoint operator $A$ (the **Hamiltonian**). So we may assume that $A = A_m$ on $H = L^2(\Omega, \Sigma, \mu)$ where $m : \Omega \to \mathbb{R}$ is measurable and $(\Omega, \Sigma, \mu)$ a finite measure space. The states of the observable $A$ are given by unit vectors $u \in H$. If the observable $A$ is in the state $u$, and if $[\alpha, \beta] \subset \mathbb{R}$ is an interval, then

$$P = \int\limits_{m^{-1}([\alpha,\beta])} |u|^2 dx$$

is the probability that a measurement takes its values in $[\alpha, \beta]$. If $u$ is an eigenvector for the eigenvalue $\lambda$, then $m^{-1}(\{\lambda\})$ has positive measure and $u$ is 0 on $\Omega \setminus m^{-1}(\{\lambda\})$. So the probability $P$ is 1. We refer to the treatise of Reed–Simon [RS72] and the classical book of von Neumann [vNe55] for further information.

## 1.8.4 $T(L^2(\Omega)) \subset L^\infty(\Omega)$ implies Hilbert-Schmidt

The following more direct proof of Theorem 1.6.2 was suggested to us by M. Haase. Assume that $\Omega \subset \mathbb{R}^n$ is open with finite Lebesgue measure $|\Omega|$. Let $T \in \mathcal{L}(L^2(\Omega), L^\infty(\Omega))$ and let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\Omega)$. Then

$$\sum_{j=1}^{\infty} \|Te_j\|^2_{L^2(\Omega)} \leq |\Omega| \, \|T\|^2 . \tag{1.9}$$

This implies that $j \circ T$ is Hilbert-Schmidt where $j : L^\infty(\Omega) \to L^2(\Omega)$ is the natural injection.

Proof of (1.9): Let $d \in \mathbb{N}$, $U = \left\{\lambda = (\lambda_j)_{j=1,\ldots,d} : \sum_{j=1}^d \lambda_j^2 \leq 1\right\}$ be the unit ball of $(\mathbb{R}^d, \|\cdot\|_2)$ and let $U_0$ be a countable dense subset of $U$. For each $\lambda \in U_0$ one has

$$\left| \left( T \sum_{j=1}^{d} \lambda_j e_j \right)(x) \right| \leq \|T\| \qquad \text{a.e.}$$

Hence there exists a null set $M \subset \Omega$ such that

$$\left|\sum_{j=1}^{d} \lambda_j (Te_j)(x)\right| = \left|\left(T \sum_{j=1}^{d} \lambda_j e_j\right)(x)\right| \leq \|T\|$$

for all $\lambda \in U_0$ whenever $x \in \Omega \setminus M$. Observe that for $y \in \mathbb{R}^d$, $\sqrt{\sum_{j=1}^{d} y_j^2} = \sup_{\lambda \in U_0} \sum_{j=1}^{d} y_j \lambda_j$. Consequently

$$\sum_{j=1}^{d} |Te_j(x)|^2 \leq \|T\|^2$$

whenever $x \in \Omega \setminus M$. Integrating over $\Omega$ implies that

$$\sum_{j=1}^{d} \|Te_j\|_{L^2(\Omega)}^2 \leq |\Omega| \, \|T\|^2 .$$

# Lecture 2

# Semigroups

In this chapter we give a short introduction to semigroups. We start with a preliminary technical section.

## 2.1 The vector-valued Riemann integral

Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $-\infty < a < b < \infty$. By $C([a,b], X)$ we denote the space of all continuous functions on $[a,b]$ with values in $X$. Let $u \in C([a,b], X)$. Let $\pi$ be a **partition** $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$ with **intermediate** points $s_i \in [t_{i-1}, t_i]$. By $|\pi| = \max\limits_{i=1,\ldots n} (t_i - t_{i-1})$ we denote the **norm** of $\pi$ and by

$$S(\pi, u) = \sum_{i=1}^{n} u(s_i)(t_i - t_{i-1})$$

the **Riemann sum** of $u$ with respect to $\pi$. One shows as in the scalar case that

$$\int_a^b u(s)\mathrm{ds} := \lim_{|\pi| \to 0} S(\pi, u) \tag{2.1}$$

exists. If $Y$ is another Banach space and $B \in \mathcal{L}(X, Y)$, then $BS(\pi, u) = S(\pi, Bu)$ where $Bu = B \circ u \in C([a,b], Y)$. It follows that

$$B \int_a^b u(s)\mathrm{ds} = \int_a^b Bu(s)\mathrm{ds}\ . \tag{2.2}$$

In particular,

$$\langle x', \int_a^b u(s)\mathrm{ds} \rangle = \int_a^b \langle x', u(s) \rangle \mathrm{ds}\ . \tag{2.3}$$

Now the Hahn-Banach theorem allows us to carry over the usual properties of scalar Riemann integral to the vector-valued case. For example, the mapping $u \mapsto \int_a^b u(t)dt$ from $C([a,b],X)$ into $X$ is linear. We also note that

$$\|\int_a^b u(s)\mathrm{ds}\| \le \int_a^b \|u(s)\|\mathrm{ds} \tag{2.4}$$

as is easy to see.

Let $A$ be a closed operator on $X$. Let $u \in C([a,b],D(A))$, where $D(A)$ is considered as a Banach space with the graph norm; i.e. $u \in C([a,b],X)$ such that $u(t) \in D(A)$ for all $t \in [a,b]$ and $Au \in C([a,b];X)$. Since $A \in \mathcal{L}(D(A),X)$, (2.2) implies that

$$A\int_a^b u(s)\mathrm{ds} = \int_a^b Au(s)\mathrm{ds}\ . \tag{2.5}$$

## 2.2 Semigroups

In this section we introduce semigroups and their generators. Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

**Definition 2.2.1.** *A $C_0$-**semigroup** is a mapping $T : \mathbb{R}_+ \to \mathcal{L}(X)$ such that*

1. *$T(\cdot)x : \mathbb{R}_+ \to X$ continuous for all $x \in X$;*
2. *$T(t+s) = T(t)T(s) \quad (s,t \in \mathbb{R}_+)$;*
3. *$T(0) = I$.*

It follows immediately from the definition that

$$T(t)T(s) = T(s)T(t) \text{ for all } t,s \ge 0\ . \tag{2.6}$$

Let $T : \mathbb{R}_+ \to \mathcal{L}(X)$ be a $C_0$-semigroup. We now define the **generator** of $T$.

**Definition 2.2.2.** *The generator $A$ of $T$ is the operator $A$ on $X$ given by*

$$\begin{aligned} D(A) &= \{x : \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists in } X\} \\ Ax &= \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)\ . \end{aligned}$$

We now investigate relations between the semigroup $T$ and its generator $A$. One has

$$T(t)x \in D(A) \text{ and } AT(t)x = T(t)Ax \tag{2.7}$$

for all $x \in D(A), t \ge 0$. In fact, $\frac{1}{h}(T(h)T(t)x - T(t)x) = T(t)[\frac{1}{h}(T(h)x - x)] \to T(t)Ax$ $(h \downarrow 0)$. This shows in particular that the right derivate of $T(t)x$ is $T(t)Ax$ if $x \in D(A)$. More is true.

**Proposition 2.2.3.** *Let $x \in D(A)$. Then $u(t) = T(t)x$ is the unique solution of the initial value problem*

$$\left\{ \begin{array}{lcl} \multicolumn{3}{l}{u \in C^1(\mathbb{R}_+, X)\ ,\ u(t) \in D(A) \quad (t \geq 0)\ ;} \\ \dot{u}(t) & = & Au(t) \quad (t \geq 0) \\ u(0) & = & x\ ; \end{array} \right. \tag{2.8}$$

*Proof.* Let $t > 0$. It follows from the uniform boundedness principle that $T$ is bounded on $[0, t]$. Then

$$\begin{aligned} \frac{1}{-h}(T(t-h)x - T(t)x) &= T(t-h)[\frac{T(h)x - x}{h}] = T(t-h)[\frac{T(h)x - x}{h} - Ax] \\ &+ T(t-h)Ax \to T(t)Ax \quad (h \downarrow 0)\ . \end{aligned}$$

This shows that $u$ is also left differentiable and indeed a solution of the problem (2.8). Conversely, let $v$ be another solution. Let $t > 0,\ w(s) = T(t-s)v(s)$. Then

$$\begin{aligned} \frac{d}{ds}w(s) &= -A(T(t-s)v(s)) + T(t-s)\dot{v}(s) \\ &= -T(t-s)Av(s) + T(t-s)Av(s) = 0\ . \end{aligned}$$

It follows that $w$ is constant. Hence $T(t)x = w(0) = w(t) = v(t)$. □

Proposition 2.2.3 shows why generators of $C_0$-semigroups are interesting. The initial value problem (2.8) has a unique solution for initial values $x$ in the domain of the generator. Moreover, the orbit $T(\cdot)x$ is the solution. There is another way to describe the generator $A$.

**Proposition 2.2.4.** *Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$ if and only if*

$$\int_0^t T(s)y\mathrm{ds} = T(t)x - x \quad (t \geq 0)\ . \tag{2.9}$$

*Proof.* Assume (2.9). Then $\lim\limits_{t\downarrow 0} \frac{T(t)x-x}{t} = \lim\limits_{t\downarrow 0} \frac{1}{t}\int_0^t T(s)y\mathrm{ds} = y$. Conversely, let $x \in D(A)$, then $T(\cdot)x$ is the solution of (2.8). By the fundamental theorem of calculus, $T(t)x - x = \int_0^t \frac{d}{dt}T(s)x\mathrm{ds} = \int_0^t T(s)Ax\mathrm{ds}$. □

**Corollary 2.2.5.** *The operator $A$ is closed.*

*Proof.* Let $x_n \in D(A), x_n \to x, y_n := Ax_n \to y\ (n \to \infty)$. Then by (2.9),

$$\int_0^t T(s)y_n\mathrm{ds} = T(t)x_n - x_n\ .$$

Letting $n \to \infty$ shows that (2.9) holds. □

Let $x \in D(A)$. Since $A$ is closed it follows from (2.9) and (2.5) that $\int_0^t T(s)x\mathrm{ds} \in D(A)$ and

$$\begin{aligned} A\int_0^t T(s)x\mathrm{ds} &= \int_0^t AT(s)x\mathrm{ds} = \int_0^t T(s)Ax\mathrm{ds} \\ &= T(t)x - x \end{aligned}$$

for all $t \geq 0$. This identity remains valid for all $x \in X$.

**Proposition 2.2.6.** *Let $x \in X,\ t \geq 0$. Then $\int_0^t T(s)x\mathrm{ds} \in D(A)$ and*

$$A\int_0^t T(s)x\mathrm{ds} = T(t)x - x. \tag{2.10}$$

*Proof.* In fact,

$$\begin{aligned} &\frac{1}{h}\{T(h)\int_0^t T(s)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}\} \\ &= \frac{1}{h}(\int_0^t T(s+h)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}) \\ &= \frac{1}{h}(\int_h^{t+h} T(s)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}) \\ &= \frac{1}{h}(\int_t^{t+h} T(s)x\mathrm{ds} - \int_0^h T(s)x\mathrm{ds}) \quad \to T(t)x - x \text{ as } h \downarrow 0\ . \end{aligned}$$

$\square$

**Corollary 2.2.7.** *The domain of $A$ is dense in $X$.*

*Proof.* Let $x \in X$. Then $\frac{1}{t}\int_0^t T(s)x\mathrm{ds} \in D(A)$ and $\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)x\mathrm{ds} = x$. $\square$

Since $A$ is closed, the space $D(A)$ is a Banach space with the graph norm. The properties shown above imply that

$$T(\cdot)x \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, X)$$

for all $x \in D(A)$. If $x \in X$, then there exist $x_n \in D(A)$ such that $x_n \to x$ $(n \to \infty)$. Then $T(t)x_n$ converges to $T(t)x$ as $n \to \infty$. Since $T(\cdot)x_n$ is a solution of the initial value problem (2.8) with initial value $x_n$ we may consider $u(t) = T(t)x$ as a "mild solution" of the abstract Cauchy problem

$$(ACP)\left\{\begin{array}{rcl} \dot{u}(t) &=& Au(t) \quad (t \geq 0) \\ u(0) &=& x\ . \end{array}\right.$$

## 2.3 Selfadjoint semigroups

Symmetric, m-dissipative operators can be transformed into diagonal operators or multiplication operators by the spectral theorem. After this transformation one can write down explicitly the corresponding semigroup. We obtain the most simple $C_0$-semigroups with unbounded generator. Still we will see in the next two chapters that many concrete examples are of this form. We first consider the case where $A$ has compact resolvent. In fact, in that case, the spectral theorem is particularly easy to prove and the operator is transformed into a diagonal operator. In addition, our prototype example is of this type, namely the Laplacian with Dirichlet boundary conditions on a bounded open set. So the functional analytic tools needed for this important example are particularly simple.

Let $H$ be a complex, separable Hilbert space and let $A$ be a symmetric, m-dissipative operator on $H$. Assume first that $A$ has compact resolvent. Then, up to unitary equivalence, we can assume that

$$H = \ell^2 \ , \ Ax = -(\lambda_n x_n)_{n\in\mathbb{N}}$$

where $\lambda_n \in \mathbb{R}_+$, $\lim\limits_{n\to\infty} \lambda_n = \infty$ and

$$D(A) = \{x \in \ell^2 : (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\} \ .$$

Define $T(t) \in \mathcal{L}(\ell^2)$ by

$$T(t)x = (e^{-\lambda_n t} x_n)_{n\in\mathbb{N}} \ . \tag{2.11}$$

Then $T(t)$ is a compact, selfadjoint operator and $\|T(t)\| \le 1$. It is easy to see that $T = (T(t))_{t\ge 0}$ is a $C_0$-semigroup and $A$ its generator.

In the general case, if the resolvent is not necessarily compact, then after a unitary transformation we can assume that

$$\begin{aligned} H &= L^2(X, \Sigma, \mu) \\ Af &= -mf \\ D(A) &= \{f \in H : m \cdot f \in H\} \end{aligned}$$

where $(X, \Sigma, \mu)$ is a finite measure space and $m : X \to \mathbb{R}_+$ a measurable function. Now it is easy to see that

$$T(t)f = e^{-tm} f \tag{2.12}$$

defines a $C_0$-semigroup of selfadjoint operators. Moreover, $\|T(t)\| \le 1$. We have proved the following result.

**Theorem 2.3.1.** *Let $A$ be a symmetric, m-dissipative operator. Then $A$ generates a $C_0$-semigroup $T$ of contractive, selfadjoint operators. If $A$ has compact resolvent, then $T(t)$ is compact for all $t > 0$.*

Applying Theorem 2.3.1 it is frequently useful to have the concrete representation (2.11) or (2.12) in mind which is valid after a unitary transformation in virtue of the Spectral Theorem. It shows for example the following simple result on the asymptotic behaviour of the semigroup.

**Corollary 2.3.2.** *Let $A$ be a symmetric, m-dissipative operator with compact resolvent. Assume that* $\ker A = \{0\}$. *Then there exists $\varepsilon > 0$ such that*

$$\|T(t)\| \leq e^{-\varepsilon t} \quad (t \geq 0) \ . \tag{2.13}$$

## 2.4 The Hille–Yosida Theorem

Given an operator $A$ it is desirable to find criteria which imply that $A$ is the generator of a $C_0$-semigroup Most characterizations are based on conditions on the resolvent of the operator. In fact, since a $C_0$-semigroup is always exponentially bounded, the Laplace transform always exists and it turns out to be the resolvent of the operator. In this section we characterize generators of contraction semigroups (Hille-Yosida Theorem). It is a real characterization in the sense that the resolvent is supposed to exist on the right half-line. We start proving exponential boundedness.

Let $X$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. Assume that $A$ generates a $C_0$-semigroup $T$. Then there exist $M \geq 0, \omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad (t \geq 0) \ . \tag{2.14}$$

*Proof of (2.14).* Let $M := \sup\limits_{0 \leq t \leq 1} \|T(t)\|$. Then $M < \infty$ by the uniform boundedness principle. Let $\omega = \log M$. Let $t \geq 0$. Take $n \in \mathbb{N}_0$ and $s \in [0, 1)$ such that $t = n + s$. Then $\|T(t)\| = \|T(s)T(1)^n\| \leq MM^n = Me^{\omega n} \leq Me^{\omega t}$. ☐

We denote by

$$\omega(A) := \inf\{\omega \in \mathbb{R} : \exists\, M \geq 0 \text{ such that (2.14) holds}\}$$

the **growth bound** of $T$.

**Proposition 2.4.1.** *If $\lambda > \omega(A)$ (resp.* $\operatorname{Re}\lambda > \omega(A)$ *if $\mathbb{K} = \mathbb{C}$), then $\lambda \in \varrho(A)$ and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt \quad (x \in X) \ .$$

*Proof.* The operator $A - \lambda$ generates the semigroup $(e^{-\lambda t}T(t))_{t\geq 0}$ (see Exercise 2.6.2). Thus by (2.10) we get $(A-\lambda)\int\limits_0^t e^{-\lambda s}T(s)x\,ds = e^{-\lambda t}T(t)x - x$. Since $A$ is closed, letting $t \to \infty$, it follows that $Rx \in D(A)$ and $(A-\lambda)Rx = -x$ where $Rx := \int\limits_0^\infty e^{-\lambda t}T(t)x\,dt$, $x \in X$. If $x \in D(A)$, then

$$R(\lambda - A)x = -\lim_{b\to\infty}\int_0^b e^{-\lambda t}T(t)(A-\lambda)x\,dt = -\lim_{b\to\infty}(e^{-\lambda b}T(b)x - x) = x.$$

We have shown that $R$ is the inverse of $(\lambda - A)$. □

Thus, if $A$ generates a $C_0$-semigroup, then the half-plane $\{\operatorname{Re}\lambda > \omega(A)\}$ is in the resolvent set (if $\mathbb{K} = \mathbb{C}$).

A contraction $C_0$-semigroup is a $C_0$-semigroup $T$ satisfying

$$\|T(t)\| \leq 1 \quad (t \geq 0).$$

**Theorem 2.4.2** (Hille–Yosida)**.** *Let $A$ be a densely defined operator on $X$. The following assertions are equivalent:*

*(i) $A$ generates a contraction $C_0$-semigroup $T$.*

*(ii) $(0,\infty) \subset \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1 \quad (\lambda > 0)$.*

*In that case*

$$T(t)x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}x \quad (x \in X) \tag{2.15}$$

(2.15) *is called* **Euler's Formula**.

## 2.5 Holomorphic Semigroups

Next we introduce holomorphic semigroups. We use a definition which is valid for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

**Definition 2.5.1.** *Let $T$ be a $C_0$-semigroup with generator $A$. We say that $T$ is* **holomorphic** *if there exists a constant $c > 0$ such that*

$$\|tAT(t)x\| \leq c\|x\| \quad (0 < t \leq 1) \tag{2.16}$$

*for all $x \in D(A)$.*

Since $D(A)$ is dense in $X$, it follows from (2.16) that $T(t)X \subset D(A)$ for $0 < t \leq 1$ and (2.16) is valid for all $x \in X$.

If $\mathbb{K} = \mathbb{R}$, then it is obvious that $A$ generates a holomorphic $C_0$-semigroup on $X$ if and only if the complexification $A_{\mathbb{C}}$ of $A$ generates holomorphic $C_0$-semigroup on the complexification $X_{\mathbb{C}}$ of $X$.

If $\mathbb{K} = \mathbb{C}$, then (2.16) can be used to extend $T$ analytically to a sector. We make this precise in the following result. For $0 < \theta < \pi$ we denote by

$$\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$$

the sector of angle $\theta$.

**Theorem 2.5.2.** *Let $T$ be a $C_0$-semigroup with generator $A$ on a complex Banach space. Then $T$ is holomorphic if and only if $T$ has a holomorphic extension $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ for some $\theta \in (0, \pi)$ which is bounded on $\Sigma_\theta \cap \{z \in \mathbb{C} : |z| \leq 1\}$.*

The following characterization is important. It is a complex characterization: we assume the resolvent to exist on a half-plane. In contrast to the Hille-Yosida Theorem no contractivity hypothesis is imposed.

**Theorem 2.5.3** (complex characterisation of generators of holomorphic semigroups)**.** *Let $A$ be a densely defined operator on a complex Banach space. The following assertions are equivalent:*

1. *$A$ generates a holomorphic $C_0$-semigroup $T$,*
2. *there exists $\omega \in \mathbb{R}$ and $M \geq 0$ such that $\lambda \in \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ whenever $\operatorname{Re}\lambda \geq \omega$.*

*In that case $\|T(t)\| \leq M' e^{\omega t} \quad (t \geq 0)$ for some $M' \geq 0$.*

If we already know that $A$ generates a $C_0$-semigroup $T$, then in *(2)* it suffices to consider one vertical line. In fact, replacing $A$ by $A - \omega_1$ for $\omega_1$ large enough, we may assume that $\|T(t)\| \leq Me^{-\omega t}(t \geq 0)$ for some $\omega > 0$ (cf. Exercise 2.6.2). Then

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt \quad (\operatorname{Re}\lambda \geq 0).$$

Hence

$$\|\lambda R(\lambda, A)\| \leq |\lambda| \int_0^\infty e^{-\operatorname{Re}(\lambda)t} dt \; M = \frac{|\lambda|}{\operatorname{Re}(\lambda)} M.$$

Thus $\lambda R(\lambda, A)$ is bounded on each sector $\Sigma_\alpha$ of angle $\alpha < \frac{\pi}{2}$.

**Corollary 2.5.4.** *Let $A$ be the generator of a $C_0$-semigroup $T$ on a complex Banach space $E$. Then $T$ is holomorphic if and only if there exists $\omega > \omega(A)$ such that*

$$\sup_{s\in\mathbb{R}} \|sR(is+\omega, A)\| < \infty \ .$$

*Proof.* We show that the condition is sufficient. Considering $A$ instead of $A-\omega$ we may assume that $\omega(A) < 0$ and $\omega = 0$. Now expanding $R(is, A)$ analytically according to Proposition 1.2.2 one sees that $\lambda \in \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ for all $\lambda \in i\Sigma_\theta$ for some $\theta \in (0, \pi/2)$. Since $\lambda R(\lambda, A)$ is bounded on $\Sigma_{\pi/2-\varepsilon}$ for all $\varepsilon > 0$, it follows that $\lambda R(\lambda, A)$ is bounded on $\Sigma_{\pi/2+\theta}$. □

There is still another most interesting criterion for holomorphy in terms of the asymptotic behaviour of $T(t)$ as $t \downarrow 0$.

**Theorem 2.5.5** (Kato–Neuberger)**.** *Let $T$ be a semigroup on a Banach space $E$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If*

$$\overline{\lim_{t\downarrow 0}} \|T(t) - I\| < 2 \ ,$$

*then $T$ is holomorphic.*

*Proof.* Considering $A - \omega$ instead of $A$ we may assume that $\omega(A) < 0$ and $\|T(t)\| \leq M \quad (t \geq 0)$. By assumption, there exist $\varepsilon > 0, t_0 > 0$ such that $\|T(t) - I\| \leq 2 - \varepsilon$ for $0 < t \leq t_0$. Hence for $x \in X, 0 < t \leq t_0$,

$$\begin{aligned} \|x + T(t)x\| &= \|2x - (x - T(t)x)\| \\ &\geq 2\|x\| - \|x - T(t)x\| \\ &\geq \varepsilon\|x\| \ . \end{aligned}$$

Let $\alpha \in \mathbb{R}, |\alpha| \geq \frac{\pi}{t_0}$. We show that $|\alpha|\|R(i\alpha, A)\| \leq M$. Let $t = \pi/|\alpha| \leq t_0$. Then for $x \in X$,

$$\begin{aligned} \varepsilon\|x\| &\leq \|T(t)x + x\| \\ &= \|e^{\pm i\alpha t}T(t)x - x\| \\ &= \|\int_0^t e^{\pm i\alpha s}T(s)(A \pm i\alpha)x ds\| \\ &\leq tM\|(A \pm i\alpha)x\| \\ &= \frac{\pi}{|\alpha|}M\|(A \pm i\alpha)x\| \ . \end{aligned}$$

This implies that

$$|\alpha|\|R(i\alpha, A)\| \leq \frac{1}{\varepsilon}M\pi \ .$$

This concludes the proof by Corollary 2.5.4. Observe that above we applied (2.9) to the operator $A \pm i\alpha$, which generates the semigroup $(e^{\pm i\alpha t}T(t))_{t\geq 0}$. □

The converse of Theorem 2.5.5 holds for contraction semigroups on uniformly convex spaces, in particular on Hilbert spaces.

**Theorem 2.5.6.** *Let $T$ be a contraction $C_0$-semigroup on a uniformly convex space. If $T$ is holomorphic then*

$$\lim_{t\downarrow 0} \|T(t) - I\| < 2 \ .$$

We conclude this section by the following characterization of compactness. Note that each holomorphic semigroup $T$ is **immediately norm-continuous** , i.e., continuous on $(0, \infty)$ with values in $\mathcal{L}(X)$.

**Proposition 2.5.7.** *Let $T$ be a $C_0$-semigroup. The following assertions are equivalent.*

*(i) $T(t)$ is compact for all $t > 0$;*

*(ii) $T$ is immediately norm continuous and its generator $A$ has compact resolvent.*

*Proof.* Replacing $A$ by $A - \omega$ we may assume that $A$ is invertible. Let $S(t) = \int_0^t T(s)ds$. Then by Proposition 2.2.6

$$S(t) = T(t)A^{-1} - A^{-1} \tag{2.17}$$

$(i) \Rightarrow (ii)$ a) Since $T(t) \to I$ as $t \downarrow 0$ strongly, $\lim_{t\downarrow 0} T(t)x = x$ uniformly in $x \in K$, if $K \subset X$ is compact. Let $t_0 > 0$. Then $T(t_0)B$ is relatively compact, where $B$ denotes the closed unit ball. Hence $\lim_{t\downarrow 0} T(t + t_0)x = \lim_{t\downarrow 0} T(t)T(t_0)x$ uniformly for $x \in B$. This shows that $T$ is right norm-continuous. Since

$$T(t_0 - t) - T(t_0) = T(\frac{t_0}{2} - t)[T(\frac{t_0}{2}) - T(\frac{t_0}{2} + t)] \ ,$$

$T$ is also left-continuous in $t_0 > 0$.
b) Since $S(t)$ is the limit in the operator norm of compact Riemann sums, it follows that $S(t)$ is compact. It follows from (2.17) that $A^{-1}$ is compact.
$(ii) \Rightarrow (i)$ Assume that $A^{-1}$ is compact. Then $S(t)$ is compact by (2.17). If $T$ is immediately norm-continuous, then also $T(t) = \lim_{h\to 0} \frac{1}{h}(S(t + h) - S(t))$ is compact for all $t > 0$. ☐

## 2.6 Exercises

In the first two exercises we establish some standard properties of semigroup. At first we show that strong continuity in 0 implies strong continuity.

**Exercise 2.6.1** ($C_0$-semigroup). *Let $T : \mathbb{R}_+ \to \mathcal{L}(X)$ be a mapping such that*

1. $\lim_{t\downarrow 0} T(t)x = x$ *for all* $x \in X$;
2. $T(t+s) = T(t)T(s)$ $(t, s \geq 0)$.

*Show that $T$ is a $C_0$-semigroup.*

The following rescaling property is used very frequently. For many purposes it allows us to consider the case where $\omega(A) < \infty$ just by replasing $A$ by $A - \omega$.

**Exercise 2.6.2** (Rescaling).
1. *Let $\omega \in \mathbb{C}$, $S(t) = e^{-\omega t}T(t)$. Show that $S$ is a $C_0$-semigroup and $A - \omega I$ its generator.*
2. *Let $\alpha > 0$, $S(t) = T(\alpha t)$. Show that $S$ is a $C_0$-semigroup and $\alpha A$ its generator.*

**Hint:** for continuity from the left: Use the uniform boundedness principle to show that $T$ is bounded on some interval $[0, \tau]$. Deduce an estimate (2.14). Let $A$ be the generator of a $C_0$-semigroup $T$.

In the next exercise we verify the diverse criteria for holomorphy in the special case of self-adjoint operators.

**Exercise 2.6.3** (holomorphy of selfadjoint semigroups). *Let $T$ be the $C_0$-semigroup generated by a symmetric, m-dissipative operator $A$ on a separable Hilbert space $H$. Use the Spectral Theorem to show that*

*(a)* $\|tAT(t)\| \leq c \quad (0 < t \leq 1)$ *for some $c \geq 0$, i.e. $T$ is holomorphic,*

*(b)* $\lim_{t\downarrow 0} \|T(t) - I\| \leq 1 < 2$,

*(c) $T$ has a holomorphic bounded extension to $\Sigma_{\pi/2}$ with values in $\mathcal{L}(H)$.*

If $A \in \mathcal{L}(X)$, then $A$ generates the $C_0$-semigroup $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ and $\|e^{tA} - I\|_{\mathcal{L}(X)} \to 0$ as $t \downarrow 0$. The next two exercises show the converse property.

**Exercise 2.6.4** (norm-continuous semigroup). *Let $T$ be a $C_0$-semigroup with generator $A$. If $\lim_{t\downarrow 0} \|T(t) - I\| = 0$, then $A$ is bounded.*

**Hint:** Use Proposition 2.4.1 to show that $\lim_{\lambda\to\infty} \|\lambda R(\lambda, A) - I\|_{\mathcal{L}(X)} = 0$. Deduce that $D(A) = X$.

**Exercise 2.6.5** (the algebraic trick). *Let $T : (0, \infty) \to \mathcal{L}(X)$ be a function such that*

$$T(s+t) = T(t)T(s) \quad t, s \geq 0 \ .$$

*Let* $L := \overline{\lim_{t\downarrow 0}} \|T(t) - I\|$.

*a) Show $L \geq 1$ or $L = 0$.*
**Hint:** $2(T(t) - I) = T(2t) - I - (T(t) - I)^2$.

*b) Assume that $L < 1$. Show that $T$ is a $C_0$-semigroup with bounded generator.*
**Hint:** *Use Exercise 2.6.5.*

## 2.7 Comments

Most of the material presented here can be found in all books on semigroups, e.g. [EN00], [Gol85], [Paz83]. A particularly short and elegant introduction is given in Kato's monograph [Kat66]. The monograph [ABHN01] treats semigroups systematically by Laplace transform methods and describes in particular asymptotic behaviour via Tauberian theorems. The monograph [Lun95] is devoted to holomorphic semigroups as was the Internet Seminar 2004/05.
The proof of Theorem 2.5.6 can be found in Pazy's book [Paz83, Chapter 2, Corollary 5.8].
Concerning vector-valued holomorphic functions as used in the statement of Theorem 2.5.2 we refer to [ABHN01, Appendix B].

The solution of Exercise 2.6.5 indicated by the hint is due to Th. Coulhon (cf. [Nag86, A.II.3]) by a result of Lotz (and Coulhon in a special case) each generator of a $C_0$-semigroup on a space is bounded (see [ABHN01, Corollary 4.3.19]).

# Lecture 3

# The Laplacian on open sets in $\mathbb{R}^{\mathbf{n}}$

In this lecture we present the Laplacian on $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set, with Dirichlet and Neumann boundary conditions. There are two selfadjoint realisations of the Laplacian on $L^2(\Omega)$. The semigroups generated by these operators govern the solution of the heat equation with these two different boundary conditions. We merely need the definition of the Sobolev space $H^1(\Omega)$ in order to prove selfadjointness. And in fact, the Dirichlet Laplacian is the prototype example which shows the power of Hilbert space methods in the Theory of Partial Differential Equations. The simple Riesz-Fréchet Lemma representing functionals on Hilbert spaces allows us to solve the elliptic equation we need to solve in order to fulfill the range condition. We also establish some order properties of $H^1(\Omega)$. They allow us to prove that the semigroups are positive. In this lecture we assume that $\mathbb{K} = \mathbb{R}$ throughout.

## 3.1 The Dirichlet and Neumann Laplacian on open sets in $\mathbb{R}^{\mathbf{n}}$

We start introducing the first order Sobolev space on an open set of $\mathbb{R}^n$. Not much more than the definition is needed to show that the Laplacian with Dirichlet or Neumann boundary conditions generates a $C_0$-semigroup.

First we introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $L^p(\Omega)$, $1 \le p \le \infty$, is understood with respect to Lebesgue measure. We define

$$\begin{aligned}
L^1_{\text{loc}}(\Omega) \;&=\; \{f : \Omega \to \mathbb{R} \ \text{ measurable } : \int_K |f(x)|\, dx < \infty \ \text{ for all compact } \ K \subset \Omega\}\ ,\\
C(\Omega) \;&:=\; \{f : \Omega \to \mathbb{R} \ \text{ continuous}\}\ ,\\
C(\bar{\Omega}) \;&:=\; \{f : \bar{\Omega} \to \mathbb{R} \ \text{ continuous}\}\ ,\\
C^k(\Omega) \;&:=\; \{f : \Omega \to \mathbb{R} : k\text{-times continuously differentiable}\}\ ,
\end{aligned}$$

where $k \in \mathbb{N}$. For $f \in C^1(\Omega)$ we let $D_j f = \frac{\partial f}{\partial x_j}$ $(j = 1, \ldots, n)$. By $C_c(\Omega)$ we denote the space of all continuous functions $f : \Omega \to \mathbb{R}$ such that the **support** $\mathrm{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}$ is a compact subset of $\Omega$. We let

$$\begin{aligned} C_c^k(\Omega) &:= C^k(\Omega) \cap C_c(\Omega) \ , \\ C^\infty(\Omega) &:= \bigcap_{k \in \mathbb{N}} C^k(\Omega) \ ; \quad \text{and by} \\ \mathcal{D}(\Omega) &:= C^\infty(\Omega) \cap C_c(\Omega) \end{aligned}$$

we denote the space of all **test functions**. Let $f \in C^1(\Omega)$, $\varphi \in C_c^1(\Omega)$. Then

$$-\int_\Omega f D_j \varphi dx = \int_\Omega D_j f \varphi dx \ . \tag{3.1}$$

We use this relation (3.1) to define weak derivatives.

**Definition 3.1.1.** *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Let $j \in \{1, \ldots, n\}$. A function $g \in L^1_{\mathrm{loc}}(\Omega)$ is called the* **weak $j$-th partial derivative** *of $f$ (in $\Omega$) if*

$$-\int_\Omega f D_j \varphi dx = \int_\Omega g \varphi dx$$

*for all $\varphi \in \mathcal{D}(\Omega)$. Then we set $D_j f := g$.*

Note that the weak $j$-th partial derivative is unique. Here we identify functions in $L^1_{\mathrm{loc}}(\Omega)$ which coincide almost everywhere. We let

$$W^1(\Omega) = \{f \in L^1_{\mathrm{loc}}(\Omega) : \text{ the weak derivative } D_j f \in L^1_{\mathrm{loc}}(\Omega) \text{ exists for all } j = 1, \ldots, n\} \ .$$

Another notation for $W^1(\Omega)$ is $W^{1,1}_{\mathrm{loc}}(\Omega)$. Note that $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for all $1 \leq p \leq \infty$. Now we define the first Sobolev space $H^1(\Omega)$ by

$$H^1(\Omega) := \{f \in L^2(\Omega) \cap W^1(\Omega) : D_j f \in L^2(\Omega) \quad \forall j = 1, \ldots, n\} \ .$$

**Proposition 3.1.2.** *The space $H^1(\Omega)$ is a separable Hilbert space for the scalar product*

$$(f \mid g)_{H^1(\Omega)} = \int_\Omega f g dx + \sum_{j=1}^n \int_\Omega D_j f D_j g dx \ .$$

*Proof.* Consider the separable Hilbert space $H = L^2(\Omega)^{n+1}$ with norm

$$\|(u_0, u_1, \ldots, u_n)\|_H^2 = \sum_{j=0}^n \int_\Omega |u_j|^2 dx \ .$$

Then $\Phi : H^1(\Omega) \to H$, $f \mapsto (f, D_1 f, \ldots, D_n f)$ is isometric and linear. Thus it suffices to show that the image of $\Phi$ is closed. Let $f_k \in H^1(\Omega)$ such that $\lim\limits_{k\to\infty} \Phi(f_k) = (f, g_1, \ldots, g_n)$ in $H$. Then $\lim\limits_{k\to\infty} f_k = f$ and $\lim\limits_{k\to\infty} D_j f_k = g_j$ in $L^2(\Omega)$ $(j = 1, \ldots, n)$. Let $\varphi \in C_c^1(\Omega)$. Then by the Dominated Convergence Theorem

$$\begin{aligned} -\int_\Omega D_j\varphi f\,dx &= \lim_{k\to\infty}\Big(-\int_\Omega D_j\varphi f_k dx\Big) \\ &= \lim_{k\to\infty}\int_\Omega \varphi D_j f_k dx \\ &= \int_\Omega \varphi g_j dx\ . \end{aligned}$$

Thus $g_j$ is the weak $j$-th partial derivative of $f$ and $\Phi(f) = (f, g_1, \ldots, g_n)$. ☐

Next we talk about Dirichlet boundary conditions. If $n \geq 2$, then $H^1(\Omega)$ is no longer a subspace of $C(\bar{\Omega})$. Thus the elements of $H^1(\Omega)$ are merely equivalence classes; we identify functions which coincide almost everywhere. In fact, in general $\partial\Omega$ will be of measure 0, so it does not make sense to talk about the restriction to $\partial\Omega$ for functions in $H^1(\Omega)$. This leads us to the following definition: The elements of the space

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}\ ,$$

i.e., the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, are considered as those functions in $H^1(\Omega)$ which satisfy Dirichlet boundary conditions in a weak form.

Later we will investigate further properties of $H_0^1(\Omega)$.

Now we want to introduce the Dirichlet Laplacian. For $f \in C^2(\Omega)$ we define the **Laplacian** $\Delta f$ by

$$\Delta f := \sum_{j=1}^n D_j^2 f\ .$$

Similarly as for the first order derivatives, we define the **weak Laplacian** as follows. Let $f \in L^1_{\text{loc}}(\Omega), g \in L^1_{\text{loc}}(\Omega)$. We say that $\Delta f = g$ **weakly**, if

$$\int_\Omega \Delta\varphi f\,dx = \int_\Omega \varphi g\,dx \tag{3.2}$$

for all $\varphi \in \mathcal{D}(\Omega)$. In that case we write

$$\Delta f = g \quad \text{weakly (on } \Omega) \qquad \text{or in } \mathcal{D}(\Omega)'\ .$$

**Remark 3.1.3.** *a) Again $g$ is unique up to a set of measure* $0$.
*b) In the language of distributions, (3.2) just means that the distribution $\Delta f$ equals $g$.*

For $f \in W^1(\Omega)$ we denote by $\mathrm{grad} f(x) = \nabla f(x) = (D_1 f(x), \ldots, D_n f(x))$ the **gradient** of $f$. For $x, y \in \mathbb{R}^n$ we denote by $xy = \sum_{j=1}^{n} x_j y_j$ the scalar product in $\mathbb{R}^n$. Similarly, for $f, g \in W^1(\Omega)$ we let $\nabla f \nabla g = \sum_{j=1}^{n} D_j f D_j g$.

**Theorem 3.1.4** (the Dirichlet Laplacian)**.** *Define the operator $A$ on $L^2(\Omega)$ by*

$$D(A) = \{f \in H_0^1(\Omega) \ : \exists g \in L^2(\Omega) \text{ such that } \Delta f = g \text{ weakly}\},$$

$$Af = \Delta f \ .$$

*Then $A$ is a selfadjoint, dissipative operator. We denote $A$ by $\Delta_\Omega^D$ and call $A$* **the Laplacian with Dirichlet boundary conditions** *or simply the* **Dirichlet Laplacian**.

*Proof.* Let $u \in D(A)$. Since $u \in H_0^1(\Omega)$, it follows that for $v \in C_c^\infty(\Omega)$,

$$\begin{aligned} (Au \mid v)_{L^2} &= (u \mid \Delta v)_{L^2} \\ &= -\int_\Omega \nabla u \nabla v \ . \end{aligned}$$

By density, we deduce that

$$(Au \mid v)_{L^2} = -\int_\Omega \nabla u \nabla v \quad \text{for all} \quad v \in H_0^1(\Omega) \ .$$

Hence $A$ is dissipative and symmetric. It remains to prove the range condition. Let $f \in L^2(\Omega)$. Then $\phi(v) = \int_\Omega f v$ defines a continuous linear form $\phi$ on $H_0^1(\Omega)$. By the Riesz-Fréchet Lemma there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega f v = (u \mid v)_{H^1} = \int_\Omega uv + \int_\Omega \nabla u \nabla v$$

for all $v \in H_0^1(\Omega)$. For $v \in C_c^\infty(\Omega)$ we obtain $\int_\Omega fv = \int_\Omega uv - \int_\Omega u \Delta v$, i.e., $f = u - \Delta u$ weakly. $\square$

Thus the operator $\Delta_\Omega^D$ generates a contractive $C_0$-semigroup $T$ of selfadjoint operators on $L^2(\Omega)$. We frequently use the symbolic notation

$$e^{t\Delta_\Omega^D} := T(t) \quad (t \geq 0) \ .$$

This semigroup governs the heat equation with Dirichlet boundary conditions. Indeed, if $f \in L^2(\Omega)$, then $u(t) = e^{t\Delta_\Omega^D} f$ is the unique solution of

$$\begin{cases} u \in C^\infty((0,\infty); L^2(\Omega)) \cap C([0,\infty); L^2(\Omega)) \ , \\ u(t) \in H_0^1(\Omega) \quad (t > 0) \ , \\ \dot{u}(t) = \Delta u(t) \quad \text{weakly} \quad (t > 0) \ , \\ u(0) = f \ . \end{cases}$$

This can be easily shown by transforming $\Delta_\Omega^D$ into a multiplication operator with the help of the Spectral Theorem.

Next we consider Neumann boundary conditions. It is remarkable that they can be defined for arbitrary open sets.

**Theorem 3.1.5** (the Neumann Laplacian)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Define the operator $A$ on $L^2(\Omega)$ by*

$$\begin{aligned} D(A) &= \{f \in H^1(\Omega) : \text{ there exists } g \in L^2(\Omega) \text{ such that} \\ &\qquad -\int_\Omega \nabla f \nabla \varphi = \int_\Omega g\varphi \text{ for all } \varphi \in H^1(\Omega)\} \\ Af &= g \ . \end{aligned}$$

*Then $A$ is selfadjoint and dissipative. We call $A$ the* **Laplacian with Neumann boundary conditions** *or simply the* **Neumann Laplacian**. *We denote the operator by $\Delta_\Omega^N$.*

The proof is similar and can be omitted.

Remark that

$$\Delta_\Omega^N f = \Delta f \quad \text{weakly}$$

for all $f \in D(\Delta_\Omega^N)$. This follows clearly from the definition.

**Remark 3.1.6** (comparison of classical and weak Neumann boundary conditions)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded with boundary of class $C^1$. Recall* **Green's Formula**

$$\int_\Omega \Delta f g dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} g \, d\sigma - \int_\Omega \nabla f \nabla g dx \tag{3.3}$$

*($f \in C^2(\bar{\Omega})$, $g \in C^1(\bar{\Omega})$), where $\sigma$ denotes the surface measure on $\partial\Omega$. By $\nu(x)$ we denote the* **exterior normal** *in each $x \in \partial\Omega$; and for $f \in C^1(\bar{\Omega})$, $\frac{\partial f}{\partial \nu}(x) = \nabla f(x) \cdot \nu(x)$ is the* **normal derivative** *of $f$ in $x \in \partial\Omega$. Now define the operator $B$ on $L^2(\Omega)$ by*

$$\begin{aligned} D(B) &= \{f \in C^2(\bar{\Omega}) : \frac{\partial f}{\partial \nu}_{|\partial\Omega} = 0\} \ , \\ Bf &= \Delta f \ . \end{aligned}$$

*Then* (a) $B \subset \Delta_\Omega^N$ *and*

(b) $C^2(\bar{\Omega}) \cap D(\Delta_\Omega^N) \subset D(B)$ .

*Proof.* a) Let $f \in D(B)$. Then by (3.3), $-(\Delta f \mid \varphi)_{L^2} = \int_\Omega \nabla f \nabla \varphi dx$ for all $\varphi \in C^1(\bar{\Omega})$. Since $\Omega$ is of class $C^1$, the space $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ (see [Bre83, Corollaire IX.8, p. 162]). Hence, going to the limit yields

$$-(\Delta f \mid \varphi)_{L^2} = \int_\Omega \nabla f \nabla \varphi$$

for all $\varphi \in H^1(\Omega)$. Hence $f \in D(\Delta_\Omega^N)$ and $\Delta_\Omega^N f = \Delta f$.
b) Let $f \in C^2(\bar{\Omega}) \cap D(\Delta_\Omega^N)$. Then

$$\int_\Omega \Delta f \varphi = -\int_\Omega \nabla f \nabla \varphi$$

for all $\varphi \in C^1(\bar{\Omega})$. Comparison with (3.3) shows that $\int_{\partial\Omega} \frac{\partial f}{\partial \nu} \varphi d\sigma = 0$ for all $\varphi \in C^1(\bar{\Omega})$. This implies $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$. □

The operator $\Delta_\Omega^N$ generates a $C_0$-semigroup $T$ on $L^2(\Omega)$. We frequently use the notation

$$e^{t\Delta_\Omega^N} := T(t) \quad (t > 0) \ .$$

This semigroup governs the heat equation with Neumann boundary conditions.

A natural question occurs: If $\Omega = \mathbb{R}^n$, then there is no boundary. So one expects that the Dirichlet and Neumann Laplacian coincide in this case. This is true indeed. One has $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ which is proved by the standard method of truncation and regularisation. And as one expects the semigroup generated by the Laplacian on $L^2(\mathbb{R}^n)$ is the **Gaussian semigroup** which is the prototype example for this course. It is given explicitely by the familiar Gaussian kernel

$$(4\pi t)^{-n/2} e^{-|x-y|^2/4t} \ .$$

Now, the Dirichlet Laplacian on $L^2(\mathbb{R}^n)$ is just the operator $\Delta_2 := \Delta_{\mathbb{R}^n}^D$ given by

$$\begin{aligned} D(\Delta_2) &= \{u \in H^1(\mathbb{R}^n) : \Delta u \in L^2(\mathbb{R}^n)\} \\ &= H^2(\mathbb{R}^n) \ , \\ \Delta_2 u &= \Delta u \ . \end{aligned}$$

We call $\Delta_2$ the Laplacian on $L^2(\mathbb{R}^n)$.

**Theorem 3.1.7.** *The operator $\Delta_2$ generates the $C_0$-semigroup $G$ on $L^2(\mathbb{R}^n)$ given by*

$$(G(t)f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy \ .$$

*$G$ is called the* **Gaussian semigroup**.

## 3.2 Order properties of $H^1(\Omega)$

In this section we establish some order properties of weak derivatives. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}$ be measurable. We define $f^+, f^-, |f| : \Omega \to \mathbb{R}$ by $f^+(x) = \max\{f(x), 0\}$, $f^- = (-f)^+$, $|f|(x) = \max\{f(x), -f(x)\}$. Observe that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Moreover, we define $\mathrm{sign} f : \Omega \to \mathbb{R}$ by

$$\mathrm{sign} f(x) = \begin{cases} 1 & \text{if} \quad f(x) > 0 \\ 0 & \text{if} \quad f(x) = 0 \\ -1 & \text{if} \quad f(x) < 0 \ . \end{cases}$$

Thus $|f| = (\mathrm{sign} f) \cdot f$. Finally, we define $\{f > k\} := \{x \in \Omega : f(x) > k\}$ and similarly $\{f < k\}, \{f \geq k\}$ and $\{f \leq k\}$ where $k \in \mathbb{R}$. Thus $\mathrm{sign} f = 1_{\{f \geq 0\}} - 1_{\{f \leq 0\}}$.

**Proposition 3.2.1.** *Let $f \in W^1(\Omega)$. Then $f^+, f^-, |f| \in W^1(\Omega)$ and*

$$D_j f^+ = 1_{\{f>0\}} D_j f \ , \tag{3.4}$$

$$D_j f^- = -1_{\{f<0\}} D_j f \ , \tag{3.5}$$

$$D_j |f| = (\mathrm{sign} f) D_j f \ . \tag{3.6}$$

*$(j = 1, \ldots, n)$. In particular, $(f - k)^+ \in W^1(\Omega)$ and $D_j(f - k)^+ = 1_{\{f>k\}} D_j f$ for all $k \in \mathbb{R}$.*

We refer to [GT98, p. 152] for the proof.

Note that the identities in Proposition 3.2.1 have to be understood in $W^1(\Omega)$, i.e., almost everywhere on $\Omega$. The first is equivalent to

$$\int_\Omega f^+(x) D_j \varphi(x) dx = \int_{\{f>0\}} D_j f(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{D}(\Omega)$.

We note the following corollary:

**Corollary 3.2.2** (Stampacchia's Lemma)**.** *Let $f \in W^1(\Omega)$, $k \in \mathbb{R}$. Then*

$$D_j f(x) = 0 \textit{ for almost all } x \in \{y \in \Omega : f(y) = k\} \ .$$

*Proof.* Replacing $f$ by $f - k$ we can assume that $k = 0$. Since $f = f^+ - f^-$ we have $D_j f = D_j f^+ - D_j f^- = 1_{\{f>0\}} D_j f - 1_{\{f<0\}} D_j f$. ☐

**Corollary 3.2.3.** *Let $f \in H^1(\Omega)$. Then $|f|, f^+, f^- \in H^1(\Omega)$. Moreover, if $k > 0$, then $(f \wedge k)(x) := \min\{f(x), k\}, (f \vee k)(x) := \max\{f(x), k\}$ define functions $f \wedge k, f \vee k \in H^1(\Omega)$ and*

$$\begin{aligned} D_j(f \wedge k) &= 1_{\{f<k\}} D_j f \ , \\ D_j(f \vee k) &= 1_{\{f>k\}} D_j f \ . \end{aligned}$$

*Proof.* It follows from Proposition 3.2.1 that $D_j|f| = \text{sign} f D_j f$, $D_j f^+ = 1_{\{f>0\}} D_j f$ and $D_j f^- = 1_{\{f<0\}} D_j f \in L^2(\Omega)$ $(j = 1, \ldots n)$. This implies that $|f|, f^+, f^- \in H^1(\Omega)$. Moreover, since $f \vee k = f + (k - f)^+$, one has

$$\begin{aligned} D_j(f \vee k) &= D_j f + 1_{\{k-f>0\}} D_j(k - f) \\ &= D_j f + 1_{\{f<k\}}(-D_j f) \\ &= 1_{\{f \geq k\}} D_j f \\ &= 1_{\{f>k\}} D_j f \end{aligned}$$

by Corollary 3.2.2. Hence $D_j(f \vee k) \in L^2(\Omega)$ $(j = 1, \ldots, n)$ and so $f \vee k \in H^1(\Omega)$. □

It follows from Proposition 3.2.1 that

$$\|f\|_{H^1(\Omega)} = \| \, |f| \, \|_{H^1(\Omega)} \tag{3.7}$$

for all $f \in H^1(\Omega)$.

**Remark 3.2.4.** *However, $H^1(\Omega)$ is not a Banach lattice, since $0 \leq f \leq g$ does not imply $\|f\|_{H^1} \leq \|g\|_{H^1}$.*

**Proposition 3.2.5.** *The mappings $f \mapsto |f|$, $f \mapsto f^+$ and $f \mapsto f^-$ are continuous from $H^1(\Omega)$ into $H^1(\Omega)$.*

For the proof we need the following well-known lemma on *weak convergence*.

**Lemma 3.2.6.** *Let $H$ be a Hilbert space. Let $x, x_k \in H$ such that $x_k \rightharpoonup x$ and $\overline{\lim_{k\to\infty}} \|x_k\| \leq \|x\|$. Then $x_k \to x$.*

Here $x_k \rightharpoonup x$ means **weak convergence** , i.e., $(x_k \mid y) \to (x \mid y)$ for all $y \in H$.

*Proof.* $\overline{\lim_{k\to\infty}} \|x - x_k\|^2 = \overline{\lim_{k\to\infty}} \{(x - x_k \mid x) - (x \mid x_k) + \|x_k\|^2\} = -\|x\|^2 + \overline{\lim_{k\to\infty}} \|x_k\|^2 = 0$. □

*Proof of Proposition 3.2.5.* Let $u_k \to u$ in $H^1$. We want to show that $|u_k| \to |u|$. For this, it suffices to show that $|u_{k_\ell}| \to |u|$ for some subsequence (since then each subsequence has a subsequence converging to $|u|$). Since $H^1$ is reflexive, taking a subsequence, we may assume that $|u_k| \rightharpoonup v$ in $H^1$. Since $|u_k| \to |u|$ in $L^2(\Omega)$, it follows that $v = |u|$. Moreover, by (3.7), $\overline{\lim_{k\to\infty}} \||u_k|\|_{H^1} = \overline{\lim_{k\to\infty}} \|u_k\|_{H^1} = \|u\|_{H^1} = \||u|\|_{H^1}$. Now the claim follows from Lemma 3.2.6. Since $u^+ = \frac{1}{2}(u + |u|)$ and $u^- = (-u)^+$, the continuity of all the three mappings is proved. □

**Corollary 3.2.7.** *Let $f \in H^1_0(\Omega)$. Then $f^+, f^-, |f| \in H^1_0(\Omega)$.*

*Proof.* a) Let $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi^+ \in H^1(\Omega)$ by Proposition 3.2.1. Since $\varphi^+$ has compact support, it follows from Proposition 3.2.8 below that $\varphi^+ \in H^1_0(\Omega)$.
b) Let $f \in H^1_0(\Omega)$. Let $f_m \in \mathcal{D}(\Omega)$ such that $f_m \to f$ in $H^1(\Omega)$ as $m \to \infty$. Then $f_m^+ \in H^1_0(\Omega)$ by a) and $\lim_{m\to\infty} f_m^+ \to f^+$ in $H^1(\Omega)$ by Proposition 3.2.5. Thus $f^+ \in H^1_0(\Omega)$. Hence also $f^- = (-f)^+$ and $|f| = f^+ + f^- \in H^1_0(\Omega)$. □

The elements of $H^1_0(\Omega)$ are considered as those functions in $H^1(\Omega)$ which vanish at the boundary in a weak sense. In the next proposition we compare the weak and the strong sense. If $\Omega \subset \mathbb{R}^n$ is open and bounded we let $C_0(\Omega) := \{f \in C(\bar{\Omega}) : f(z) = 0$ for all $z \in \partial\Omega\}$, where $\partial\Omega$ denotes the boundary of $\Omega$.

**Proposition 3.2.8.** *a) If $u \in H^1(\Omega)$ vanishes outside a compact set $K \subset \Omega$, then $u \in H^1_0(\Omega)$.*
*b) If $\Omega$ is bounded, then $C_0(\Omega) \cap H^1(\Omega) \subset H^1_0(\Omega)$.*

*Proof.* a) Denote by $(\varrho_k)_{k\in\mathbb{N}}$ a mollifier, i.e. $0 \le \varrho_k \in \mathcal{D}(\mathbb{R}^n), \int \varrho_k = 1, \operatorname{supp}\varrho_k \subset B(0,1/k)$. Then $u * \varrho_k \in \mathcal{D}(\Omega)$ for $k$ large enough and $u * \varrho_k \to u$ in $L^2(\Omega)$ and $D_j(u * \varrho_k) = D_j u * \varrho_k \to D_j u$ in $L^2(\Omega)$ as $k \to \infty$.
b) Let $u \in C_0(\Omega) \cap H^1(\Omega)$. For $k \in \mathbb{N}$ let $u_k = (u - \frac{1}{k})^+$. Then $u_k \in H^1(\Omega)$ by Proposition 3.2.1. Since $u_k$ vanishes outside of a compact set, it follows that $u_k \in H^1_0(\Omega)$ by a). Moreover, $u_k \to u^+$ in $L^2(\Omega)$ and $D_j u_k = 1_{\{u > \frac{1}{k}\}} D_j u \to D_j u$ a.e. since by Stampacchia's Lemma $D_j u = 0$ a.e. on $\{u = 0\}$. The Dominated Convergence Theorem implies that $D_j u_k \to D_j u$ in $L^2(\Omega)$ as $k \to \infty$. □

## 3.3 Positivity

The aim of this section is to show that the semigroups generated by the Dirichlet and Neumann Laplacian are positive.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \le p \le \infty$. By $L^p(\Omega)_+ = \{f \in L^p(\Omega) : f \ge 0\}$ we denote the **positive cone** in $L^p(\Omega)$ (where $f \ge 0$ means that $f(x) \in \mathbb{R}_+$ a.e.). A bounded operator $B$ on $L^p(\Omega)$ is called **positive** (we write $B \ge 0$) if $BL^p(\Omega)_+ \subset L^p(\Omega)_+$. Finally, a $C_0$-semigroup $T$ on $L^p(\Omega)$ is called **positive** if $T(t) \ge 0$ for all $t \ge 0$.

Now we consider the $C_0$-semigroups $(e^{t\Delta_\Omega^D})_{t\ge0}$ and $(e^{t\Delta_\Omega^N})_{t\ge0}$ on $L^2(\Omega)$ generated by the Dirichlet Laplacian $\Delta_\Omega^D$ and the Neumann Laplacian $\Delta_\Omega^N$, respectively.

**Theorem 3.3.1.** *The semigroups $(e^{t\Delta_\Omega^D})_{t\ge0}$ and $(e^{t\Delta_\Omega^N})_{t\ge0}$ on $L^2(\Omega)$ are positive.*

*Proof.* Let $A = \Delta_\Omega^D$ or $\Delta_\Omega^N$, $T(t) = e^{tA}$. Since by Euler's formula,

$$e^{tA} = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n}$$

strongly, it suffices to show that $R(\lambda, A) \geq 0$ for $\lambda > 0$. Let $A = \Delta_\Omega^D$ or $A = \Delta_\Omega^N$. Let $0 \geq f \in L^2(\Omega)$, $u = R(\lambda, A)f$. We have to show that $u \leq 0$. One has

$$\begin{aligned}
\lambda \|u^+\|_{L^2}^2 &= (\lambda u \mid u^+) \\
&= (\lambda u - Au \mid u^+) + (Au \mid u^+) \\
&= (f \mid u^+) + (Au \mid u^+) \\
&\leq (Au \mid u^+) && (\text{since } \ f \leq 0) \\
&= -\sum_{j=1}^n \int_\Omega D_j u D_j u^+ dx && (\text{by the definition of } A) \\
&= -\sum_{j=1}^n \int_\Omega (D_j u^+)^2 dx && (\text{by (3.4)}) \\
&\leq 0 \ .
\end{aligned}$$

Hence $u^+ = 0$; i.e., $u \leq 0$. □

## 3.4 The Poincaré Inequality and exponential stability

Let $\Omega$ be an open set in $\mathbb{R}^n$. We say that $\Omega$ lies in a **strip** of width $d$ if there exists $j_0 \in \{1, \ldots, n\}$ such that $|x_{j_0}| \leq d$ for all $x = (x_1, \ldots, x_n) \in \Omega$.

**Theorem 3.4.1** (Poincaré's Inequality). *Let $\Omega \subset \mathbb{R}^n$ be open and assume that it lies in a strip of width $d$. Then for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq 2d \left( \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}} .$$

*Proof.* By density, it suffices to show the inequality for $u \in \mathcal{D}(\Omega)$. Let us without loss of generality assume that $j_0 = 1$, i.e., $|x_1| \leq d$ for all $x = (x_1, \ldots x_n) \in \Omega$.

a) Let $h \in C^1[-d, d]$, $h(0) = 0$. Then by the Cauchy–Schwarz inequality

$$\begin{aligned}
\int_{-d}^d |h(x)|^2 dx &= \int_{-d}^d \left| \int_{-d}^x h'(y) dy \right|^2 dx \\
&\leq \int_{-d}^d \left( \int_{-d}^x |h'(y)|^2 dy \right) (x + d) dx \\
&\leq 2d \int_{-d}^d \int_{-d}^x |h'(y)|^2 dy dx \\
&= (2d)^2 \int_{-d}^d |h'(y)|^2 dy.
\end{aligned}$$

b) Let now $u \in \mathcal{D}(\Omega)$. Then by a) one has

$$\begin{aligned}\int_\Omega |u|^2 &\leq (2d)^2 \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \int_{-d}^{d} |D_1 u(x_1, \dots, x_n)|^2 dx_1 \dots dx_n \\ &\leq (2d)^2 \int_\Omega |\nabla u|^2.\end{aligned}$$

This concludes the proof. □

Poincaré's inequality implies that

$$[u \mid v] := \int_\Omega \nabla u \nabla v$$

defines an equivalent scalar product on $H_0^1(\Omega)$ if $\Omega$ lies in a strip.

Now we want to reformulate Poincaré's inequality in terms of the asymptotic behaviour of the semigroup $(e^{t\Delta_\Omega^D})_{t\geq 0}$ on $L^2(\Omega)$.

**Proposition 3.4.2.** *Let $A$ be a selfadjoint operator on a Hilbert space $H$ such that*

$$\omega := \sup\{(Au \mid u)_H : u \in D(A),\ \|u\|_H = 1\} < \infty.$$

*Then $A$ generates a $C_0$-semigroup and*

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq e^{t\omega}.$$

*Proof.* Let $\omega_1 \in \mathbb{R}$ such that $\omega_1 \geq \omega$. Then $A - \omega_1$ is dissipative, hence

$$\|e^{-t\omega_1} e^{tA}\|_{\mathcal{L}(H)} = \|e^{t(A-\omega_1)}\|_{\mathcal{L}(H)} \leq 1.$$

□

**Theorem 3.4.3.** *Let $\Omega \subset \mathbb{R}^n$ be open. If $\Omega$ lies in a strip, then there exists $\epsilon > 0$ such that*

$$\|e^{t\Delta_\Omega^D}\|_{\mathcal{L}(H)} \leq e^{-\epsilon t} \qquad (t \geq 0).$$

*Proof.* By Poincaré's inequality there exists a constant $\omega > 0$ such that $\omega \|u\|^2_{L^2(\Omega)} \leq \int_\Omega |\nabla u|^2$. Hence

$$(\Delta_\Omega^D u \mid u)_{L^2(\Omega)} = -\int_\Omega |\nabla u|^2 \leq -\omega \|u\|^2_{L^2(\Omega)}$$

for all $u \in D(\Delta_\Omega^D) \subset H_0^1(\Omega)$. □

## 3.5 Exercises

A closed subspace $J$ of $H^1(\Omega)$ is called an **ideal** if
a) $0 \le u \le v, v \in J, u \in H^1(\Omega)$ implies $u \in J$ and
b) $u \in J$ implies $|u| \in J$.

In the first exercise we show that $H_0^1(\Omega)$ is an ideal of $H^1(\Omega)$.

**Exercise 3.5.1** (ideal property of $H_0^1(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open.*
*a) Let $u, v \in H^1(\Omega), 0 \le u \le v$. Assume that $v \in H_0^1(\Omega)$. Show that also $u \in H_0^1(\Omega)$.*
*b) Show by an example that the* **order interval** $[0, v] := \{u \in H^1(\Omega) : 0 \le u \le v\}$ *is not norm bounded.*
*c) Let $u \in H_0^1(\Omega), k > 0$. Show that $u \wedge k$ and $(u-k)^+ \in H_0^1(\Omega)$.*

**Exercise 3.5.2** (points do no matter in dimension $n \ge 3$)**.** *a) Let $u \in C_c^\infty(\mathbb{R}^n), n \ge 3$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable such that $0 \le \eta \le 1$, $\eta(x) = 0$ if $|x| \le 1$ and $\eta(x) = 1$ if $|x| \ge 2$. Let $u_k(x) = \eta(k|x|)u(x)$. Show that $u_k \to u \quad (k \to \infty)$ in $H^1(\mathbb{R}^n)$.*
*b) Let $\tilde{\Omega} = \{x \in \mathbb{R}^3 : |x| < 1\}$, $\Omega = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$. Show that $\mathcal{D}(\tilde{\Omega}) \subset H_0^1(\Omega)$.*

For the next exercise we need a more precise description of the Sobolev space in dimension 1. Let $-\infty < a < b < \infty$. We say that $u \in L^2(a, b)$ is **continuous** if there exists $\tilde{u} \in C[a, b]$ such that $u = \tilde{u}$. a.e. Such $\tilde{u}$ is then unique and we identify $u$ and $\tilde{u}$. We have the following result.

**Theorem 3.5.3** (Sobolev space in 1 dimension)**.** *For $u \in L^2(a, b)$ the following are equivalent.*

*(i) $u \in H^1(a, b)$;*

*(ii) $u$ is continuous and there exists $u' \in L^2(a, b)$ such that*

$$u(t) = u(a) + \int_a^t u'(s)ds \ .$$

*In that case $u'$ is the weak derivative of $u$.*

From this we obtain the following formula for integration by parts as an application of Fubini's Theorem.

**Exercise 3.5.4** (integration by parts and the Neumann Laplacian)**.** *a) Show that*

$$\int_a^b u'(t)v(t)dt = [u(t)v(t)]_a^b - \int_a^b u(t)v'(t)dt$$

*for all $u, v \in H^1(a, b)$.*
*b) Let $\Delta_\Omega^N$ be the Neumann Laplacian on $L^2(a, b)$. Show that $\Delta_\Omega^N u = u''$ and*

$$D(\Delta_\Omega^N) = \{u \in H^2(a, b) : u'(a) = u'(b) = 0\} \ .$$

**Hint:** Observe first that $H^2(a, b) \subset C^1[a, b]$.

**Exercise 3.5.5.** *Let $\Omega$ be an open set and $u \in H_0^1(\Omega)$. Define*

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega \ , \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega \ . \end{cases}$$

*Show that $\tilde{u} \in H^1(\mathbb{R}^n)$ and $D_j\tilde{u} = (D_j u)^\sim, j = 1, \cdots n$.*

## 3.6 Comments

More information on the Dirichlet Laplacian can be found in the ISEM-Manuscript 1999 [ISEM 99/00]. The proof of Theorem 3.1.5 is given in [ISEM 99/00, Theorem 4.2.5].

Exercise 3.5.2 b) shows that in general for bounded open sets it is not true that $H_0^1(\Omega) \cap C(\bar{\Omega})$ is included in $C_0(\Omega)$. However it is true if $\Omega$ satisfies mild boundary conditions, for instance if $\Omega$ has Lipschitz boundary. In fact, recently M. Biegert and M. Warma [BW02] proved the following.

**Theorem 3.6.1.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. The following are equivalent.*

*(i)* $C(\bar{\Omega}) \cap H_0^1(\Omega) \subset C_0(\Omega)$;

*(ii)* $\Omega$ *is regular in capacity, i.e.,* $\operatorname{cap}(B(z,r) \setminus \Omega) > 0$ *for all* $z \in \partial\Omega, r > 0$.

Here $B(z,r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ denotes the ball of centre $z$ and radius $r > 0$. Moreover,

$$\operatorname{cap}(A) := \inf\{ \begin{array}{l} \|u\|_{H^1(\mathbb{R}^n)} : u \in H^1(\mathbb{R}^n)\ , \\ u \geq 1 \quad \text{in a neighborhood of} \quad A\} \end{array}$$

denotes the capacity of a subset $A$ of $\mathbb{R}^n$. This is an outer measure which allows one to describe fine properties of the Laplacian.

# Lecture 4

# Domination, Kernels and Extrapolation

In this lecture we first establish the Dunford–Pettis criterion for kernels. We apply it to the Dirichlet Laplacian. For that we will show that the semigroup generated by the Dirichlet Laplacian on $L^2(\Omega)$ is monotone as a function of the domain $\Omega \subset \mathbb{R}^n$. This is the first instance of a Gaussian estimate which will be studied more in detail later. These estimates allow us to extrapolate the semigroups to $L^p$-spaces and we investigate which properties on $L^2$ extrapolate to $L^p$.

Throughout this lecture we consider real spaces.

## 4.1 Kernels

In this section we establish the important Dunford–Pettis criterion for kernels. Let $\Omega \subset \mathbb{R}^n$ be an open set and $k \in L^\infty(\Omega \times \Omega)$. Then

$$(B_k f)(x) := \int_\Omega k(x,y) f(y) dy \tag{4.1}$$

defines a bounded operator $B_k \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ and

$$\|B_k\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \leq \|k\|_{L^\infty(\Omega\times\Omega)} \ .$$

If $E \subset \mathbb{R}^n$ is a Borel set we denote by $|E|$ the Lebesgue measure of $E$.

**Theorem 4.1.1** (Dunford–Pettis)**.** *The mapping $k \mapsto B_k$ is an isometric isomorphism from $L^\infty(\Omega \times \Omega)$ onto $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover*

$$B_k \geq 0 \text{ if and only if } k \geq 0 \tag{4.2}$$

*for all $k \in L^\infty(\Omega \times \Omega)$.*

*Proof.* For $f, g \in L^1(\Omega)$ we define $f \otimes g \in L^1(\Omega \times \Omega)$ by $(f \otimes g)(x, y) := f(x)g(y)$. Then $\|f \otimes g\|_{L^1(\Omega\times\Omega)} = \|f\|_{L^1(\Omega)} \cdot \|g\|_{L^1(\Omega)}$. It follows from the construction of the product measure that the space

$$F := \left\{ \sum_{i=1}^{n} c_i 1_{E_i} \otimes 1_{F_i} : n \in \mathbb{N}\ ,\ c_i \in \mathbb{R}\ ,\ E_i, F_i \subset \Omega \text{ measurable of finite measure} \right\}$$

is dense in $L^1(\Omega \times \Omega)$. Let $B \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Define $\phi : F \to \mathbb{R}$ by

$$\phi(u) = \sum_{i=1}^{m} c_i \int_\Omega (B1_{E_i})(y) \cdot 1_{F_i}(y) dy$$

where $u = \sum\limits_{i=1}^{m} c_i 1_{E_i} \otimes 1_{F_i}$. It is easy to see that this definition is independent of the representation of $u$ (Exercise 4.5.2). Hence $\phi : F \to \mathbb{R}$ is a linear mapping. We show that

$$|\phi(u)| \le \|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \cdot \|u\|_{L^1(\Omega\times\Omega)} .$$

For that we can assume that $(E_i \times F_i) \cap (E_j \times F_j) = \emptyset$ for $i \neq j$. This implies that

$$\|u\|_{L^1(\Omega\times\Omega)} = \sum_{i=1}^{m} |c_i| \cdot |E_i| \cdot |F_i| .$$

$$\begin{aligned}
\text{Hence } |\phi(u)| &\le \sum_{i=1}^{m} |c_i|\ \|B1_{E_i}\|_{L^\infty(\Omega)} \|1_{F_i}\|_{L^1(\Omega)} \\
&\le \sum_{i=1}^{m} |c_i|\ \|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \|1_{E_i}\|_{L^1(\Omega)}\ \|1_{F_i}\|_{L^1(\Omega)} \\
&= \|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \|u\|_{L^1(\Omega\times\Omega)} .
\end{aligned}$$

Since $(L^1(\Omega \times \Omega))' = L^\infty(\Omega \times \Omega)$, there exists a function $k \in L^\infty(\Omega \times \Omega)$ such that $\|k\|_{L^\infty(\Omega\times\Omega)} \le \|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))}$ and

$$\phi(u) = \int_\Omega \int_\Omega u(y, x) k(x, y) dy dx$$

for all $u \in F$. In particular, for simple functions $f, g \in L^1(\Omega)$ we have

$$\begin{aligned}
\int_\Omega (Bf)(x)g(x)dx = \phi(f \otimes g) &= \int_\Omega \int_\Omega k(x, y) f(y) dy g(x) dx \\
&= \int_\Omega (B_k f)(x) g(x) dx\ .
\end{aligned}$$

It follows that $Bf = B_k f$ for all simple functions $f$ in $L^1(\Omega)$. Hence $B = B_k$. We have shown that the mapping $k \mapsto B_k : L^\infty(\Omega \times \Omega) \to \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ is surjective and isometric. Finally, since functions of the type

$$u = \sum_{j=1}^{m} f_j \otimes g_j \text{ with } f_j, g_j \in L^1(\Omega)_+$$

are dense in $L^1(\Omega \times \Omega)_+$ it follows that $B_k \geq 0$ if and only if $\int_{\Omega\times\Omega} uk \geq 0$ for all $u \in L^1(\Omega \times \Omega)_+$; i.e., if and only if $k \geq 0$ a.e. . ☐

Let $B \in \mathcal{L}(L^p(\Omega))$ where $1 \leq p < \infty$. We define

$$\|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} := \sup\left\{\|Bf\|_{L^\infty(\Omega)} : f \in L^1(\Omega) \cap L^p(\Omega),\ \|f\|_{L^1(\Omega)} \leq 1\right\}.$$

Observe that $L^1(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$. To say that $\|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} < \infty$ means that there exists a unique operator $\tilde{B} \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ such that

$$\tilde{B}f = Bf \quad \text{ for all } f \in L^p(\Omega).$$

Hence by the Dunford-Pettis Theorem there exists a unique $k \in L^\infty(\Omega \times \Omega)$ such that $\tilde{B} = B_k$. We have shown the following.

**Corollary 4.1.2.** *Let $1 \leq p < \infty$, $B \in \mathcal{L}(L^p(\Omega))$ such that*

$$\|B\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} < \infty. \tag{4.3}$$

*Then there exists a function $k \in L^\infty(\Omega \times \Omega)$ such that*

$$(Bf)(x) = \int_\Omega k(x,y)f(y)dy \ a.e. \tag{4.4}$$

*for all $f \in L^1(\Omega) \cap L^p(\Omega)$. In that case $B \geq 0$ if and only if $k \geq 0$.*

We call $k$ **the kernel of** $B$. It is worth to state explicitly the following (obvious) domination property.

**Corollary 4.1.3.** *Let $1 \leq p < \infty$, $B_1, B_2 \in \mathcal{L}(L^p(\Omega))$ such that $0 \leq B_1 \leq B_2$. Assume that $B_2$ has a bounded kernel $k_2$. Then $B_1$ has a bounded kernel $k_1 \in L^\infty(\Omega \times \Omega)$ and*

$$0 \leq k_1(x,y) \leq k_2(x,y) \quad a.e.\ .$$

## 4.2 Monotonicity

Our next aim is to show that the semigroups generated by the Dirichlet-Laplacian $\Delta_\Omega^D$ and the Neumann Laplacian $\Delta_\Omega^N$ possess a kernel and that the kernels of the Dirichlet Laplacian are monotonic with respect to $\Omega$.

For an open set $\Omega \subset \mathbb{R}^n$ we consider $L^p(\Omega)$ as a subspace of $L^p(\mathbb{R}^n)$ identifying a function $f \in L^p(\Omega)$ with its extension by zero $\tilde{f} \in L^p(\mathbb{R}^n)$. If $B$ is a bounded operator on $L^p(\Omega)$ we may extend $B$ to $L^p(\mathbb{R}^n)$ by defining

$$Bf := B(f1_\Omega)\tilde{} \qquad (f \in L^p(\mathbb{R}^n)).$$

In that way $\mathcal{L}(L^p(\Omega))$ becomes a subspace of $\mathcal{L}(L^p(\mathbb{R}^n))$ such that $B \geq 0$ in $\mathcal{L}(L^p(\Omega))$ if and only if $B \geq 0$ in $\mathcal{L}(L^p(\mathbb{R}^n))$.

If $T$ is a semigroup on $L^p(\Omega)$, considering $T(t)$ as operator on $L^p(\mathbb{R}^n)$, the semigroup property

$$T(t+s) = T(t)T(s) \quad (t, s \geq 0)$$

still holds. But $T(0)$ is the projection onto $L^p(\Omega)$ given by $T(0)f = 1_\Omega f$. Moreover, the mapping $t \mapsto T(t) : \mathbb{R}_+ \to \mathcal{L}(L^p(\mathbb{R}^n))$ is strongly continuous.

If $B_1, B_2$ are bounded operators on $L^2(\mathbb{R}^n)$ we write $B_1 \leq B_2$ if $B_2 - B_1 \geq 0$. Our aim is to prove the following comparison result.

**Theorem 4.2.1.**

1. *One has always*

$$0 \leq e^{t\Delta_\Omega^D} \leq e^{t\Delta_\Omega^N}. \tag{4.5}$$

2. *If $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are open such that $\Omega_1 \subset \Omega_2$, then*

$$0 \leq e^{t\Delta_{\Omega_1}^D} \leq e^{t\Delta_{\Omega_2}^D}. \tag{4.6}$$

For the proof we use the notion of positive distributions. By $\mathcal{D}(\Omega)'$ we denote the space of all distributions. For $u \in \mathcal{D}(\Omega)'$ we write

$$u \geq 0 \text{ if } u(\varphi) \geq 0 \text{ for all } \varphi \in \mathcal{D}(\Omega)_+ .$$

Here $\mathcal{D}(\Omega)_+ := \{\varphi \in \mathcal{D}(\Omega) : \varphi \geq 0\}$. We identify $L^1_{\text{loc}}(\Omega)$ with a subspace of $\mathcal{D}(\Omega)'$ by defining $u_f \in \mathcal{D}(\Omega)'$ by

$$u_f(\varphi) := \int_\Omega f\varphi dx \quad (\varphi \in \mathcal{D}(\Omega))$$

whenever $f \in L^1_{\text{loc}}(\Omega)$. Then

$$u_f \geq 0 \text{ if and only if } f \geq 0.$$

If $u \in \mathcal{D}(\Omega)'$ the Laplacian $\Delta u \in \mathcal{D}(\Omega)'$ is defined by

$$(\Delta u)(\varphi) := u(\Delta\varphi) \quad (\varphi \in \mathcal{D}(\Omega)) \ .$$

For $u, v \in \mathcal{D}(\Omega)'$ we write

$$u \leq v \text{ if and only if } u(\varphi) \leq v(\varphi) \ \text{ for all } \varphi \in \mathcal{D}(\Omega)_+ \ .$$

Moreover, we let $H^1(\Omega)_+ := L^2(\Omega)_+ \cap H^1(\Omega)$ and $H^1_0(\Omega)_+ := L^2(\Omega)_+ \cap H^1_0(\Omega)$.

**Lemma 4.2.2.** *The cone $\mathcal{D}(\Omega)_+$ is dense in $H^1_0(\Omega)_+$.*

*Proof.* Let $f \in H^1_0(\Omega)_+$. There exist $\varphi_k \in \mathcal{D}(\Omega)$ such that $\varphi_k \to f$ in $H^1(\Omega)$ as $k \to \infty$. Hence $\varphi_k^+ \to f$ in $H^1(\Omega)$ by Proposition 3.2.5. Since $\varphi_k^+$ vanishes outside a compact set it can be approximated by positive test functions. This is done by convolving with a mollifier as in Proposition 3.2.8. ☐

**Lemma 4.2.3.** *Let $\lambda > 0$, $u \in H^1_0(\Omega)$, $0 \leq v \in H^1(\Omega)$ such that*

$$\lambda u - \Delta u \leq \lambda v - \Delta v \quad \text{ in } \mathcal{D}(\Omega)'.$$

*Then $u \leq v$.*

*Proof.* Let $0 \leq \varphi \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega \lambda u\varphi dx + \int_\Omega \nabla u \nabla\varphi dx \leq \lambda \int_\Omega v\varphi dx + \int_\Omega \nabla v \nabla \varphi dx \tag{4.7}$$

for all $0 \leq \varphi \in \mathcal{D}(\Omega)$. It follows by density that (4.7) remains true for all $\varphi \in H^1_0(\Omega)_+$ (see Lemma 4.2.2). Note that $(u-v)^+ \in H^1_0(\Omega)$. In fact, let $u_k \in \mathcal{D}(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$ as $k \to \infty$. Then $(u_k - v)^+$ has compact support, hence $(u_k - v)^+ \in H^1_0(\Omega)$ by Proposition 3.2.8. It follows that $(u-v)^+ = \lim_{k\to\infty}(u_k - v)^+ \in H^1_0(\Omega)$. Now it follows from (4.7) applied to $\varphi := (u-v)^+$ that

$$\int_\Omega \lambda u(u-v)^+ dx + \int_\Omega \nabla u \nabla (u-v)^+ dx \leq \lambda \int_\Omega v(u-v)^+ dx + \int_\Omega \nabla v \nabla (u-v)^+ dx \ .$$

Hence

$$\begin{aligned}
\int_\Omega \lambda (u-v)^{+2} dx &= \int_\Omega \lambda (u-v)(u-v)^+ dx \\
&\leq \int_\Omega \nabla (v-u) \nabla (u-v)^+ \\
&= -\int_\Omega |\nabla (u-v)^+|^2 dx \quad \text{(by (3.4))} \\
&\leq 0 \ .
\end{aligned}$$

It follows that $(u-v)^+ \leq 0$; i.e., $u \leq v$. ☐

*Proof of Theorem 4.2.1.*

1. Since $e^{tA} = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}$ strongly, where $A = \Delta_\Omega^D$ or $A = \Delta_\Omega^N$, it suffices to show that
$$R(\lambda, \Delta_\Omega^D) \le R(\lambda, \Delta_\Omega^N) \quad (\lambda > 0).$$
Let $0 < \lambda$, $u := R(\lambda, \Delta_\Omega^D)f \in H_0^1(\Omega)_+$ and $v := R(\lambda, \Delta_\Omega^N)f \in H^1(\Omega)_+$ where $f \in L^2(\Omega)_+$. Then
$$\lambda u - \Delta u = f = \lambda v - \Delta v \text{ in } \mathcal{D}(\Omega)'.$$
It follows from Lemma 4.2.3 that $u \le v$.

2. Let $\lambda > 0$, $0 \le f \in L^2(\Omega_1)$. We have to show that
$$u := R(\lambda, \Delta_{\Omega_1}^D)f \le R(\lambda, \Delta_{\Omega_2}^D)f =: v.$$
One has $u \in H_0^1(\Omega_1)_+$, $v_{|\Omega_1} \in H^1(\Omega_1)$ and $\lambda u - \Delta u = f = \lambda v - \Delta v$ weakly. It follows from Lemma 4.2.3 that $u \le v$. $\square$

It follows from Theorem 4.2.1 that

$$0 \le e^{t\Delta_\Omega^D} \le G(t) \quad (t \ge 0) \tag{4.8}$$

where $G$ denotes the Gaussian semigroup on $L^2(\mathbb{R}^n)$. From this we can deduce that $e^{t\Delta_\Omega^D}$ is a kernel operator by the the Dunford–Pettis Theorem.

We apply the preceding results to the semigroup generated by the Dirichlet Laplacian. Indeed we have the following

**Theorem 4.2.4.** *Let $\Omega \subset \mathbb{R}^n$ be open. Then $e^{t\Delta_\Omega^D}$ has a bounded kernel $k_t$ satisfying*
$$0 \le k_t(x, y) \le (4\pi t)^{-n/2} e^{-|x-y|^2/4t} \quad a.e.$$
*for all $t > 0$.*

*Proof.* It follows from Theorem 4.2.1 that $0 \le e^{t\Delta_\Omega^D} \le G(t)$. The operator $G(t)$ has the bounded kernel $(4\pi t)^{-n/2}e^{-|x-y|^2/4t}$. So the claim follows from Corollary 4.1.3. $\square$

Theorem 4.2.4 is a remarkable result. It is impossible to describe the semigroup $e^{t\Delta_\Omega^D}$ (unless $\Omega$ is of very special nature). Nonetheless we know that the semigroup has a kernel which is dominated by the Gaussian kernel. Later we will see that a quite similar result is true for a very general class of elliptic operators.

**Corollary 4.2.5.** *Let $\Omega \subset \mathbb{R}^n$ be open of finite measure. Then the operator $e^{t\Delta_\Omega^D}$ is compact for every $t > 0$. Consequently, $\Delta_\Omega^D$ has compact resolvent.*

*Proof.* Since $e^{t\Delta_\Omega^D}$ has a bounded kernel $k_t \in L^\infty(\Omega \times \Omega) \subset L^2(\Omega \times \Omega)$, it is a Hilbert-Schmidt operator (Theorem 1.6.1). Thus $e^{t\Delta_\Omega^D}$ is compact for $t > 0$. Consequently, $R(\lambda, \Delta_\Omega^D)$ is compact for $\lambda > 0$ (see Proposition 2.5.7). $\square$

## 4.3 Submarkovian Operators

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, $1 \le p \le \infty$, and let $S : L^p(\Omega) \to L^p(\Omega)$ be linear. We say that $S$ is **submarkovian** if $f \le 1$ implies $Sf \le 1$ for all $f \in L^p(\Omega)$. Here we use the notation $f \le g :\Longleftrightarrow f(x) \le g(x)$ a.e. .

**Proposition 4.3.1.** *Each submarkovian mapping is positive and hence continuous.*

*Proof.* Let $f \le 0$. Then $kf \le 1$ for all $k \in \mathbb{N}$. Hence $kSf = S(kf) \le 1$ for all $k \in \mathbb{N}$. This implies that $Sf \le 0$. We have shown that $S$ is positive. See the notes 4.6.8 for automatic continuity of positive linear mappings. $\square$

Recall that

$$L^1(\Omega) \cap L^\infty(\Omega) \subset L^p(\Omega) \subset L^1(\Omega) + L^\infty(\Omega) \tag{4.9}$$

for all $1 \le p \le \infty$.

**Lemma 4.3.2.** *Let* $S : L^1(\Omega) \cap L^\infty(\Omega) \to L^1(\Omega) + L^\infty(\Omega)$ *be positive and linear,* $1 < p < \infty$ *and let* $1/p + 1/p' = 1$. *Then for* $0 \le f, g \in L^1(\Omega) \cap L^\infty(\Omega)$,

$$S(f \cdot g) \le (Sf^p)^{1/p} \cdot (Sg^{p'})^{1/p'}. \tag{4.10}$$

*Proof.* For $a, b > 0$ one has

$$a \cdot b = \inf_{t>0} \left\{ t^p \frac{a^p}{p} + t^{-p'} \frac{b^{p'}}{p'} \right\}. \tag{4.11}$$

Hence

$$f \cdot g \le \frac{t^p}{p} f^p + \frac{t^{-p'}}{p'} g^{p'} \quad \text{a.e.} \tag{4.12}$$

and so

$$S(f \cdot g) \le \frac{t^p}{p} S(f^p) + \frac{t^{-p'}}{p'} S(g^{p'}) \quad \text{a.e.}$$

for all $t > 0$. Taking the infinum over $t > 0$ and applying (4.11) one deduces (4.10). $\square$

Let $1 \le p \le \infty$, $D \subset L^p(\Omega)$. For a linear mapping $S : D \to L^1(\Omega) + L^\infty(\Omega)$ we let

$$\|S\|_{\mathcal{L}(L^p(\Omega))} := \sup \left\{ \|Sf\|_{L^p(\Omega)} : f \in D, \|f\|_{L^p(\Omega)} \le 1 \right\}.$$

To say that an operator $S \in \mathcal{L}(L^p(\Omega))$ is submarkovian means $\|S\|_{\mathcal{L}(L^\infty(\Omega))} \le 1$ and $S \ge 0$.

**Theorem 4.3.3.** *Let* $S \in \mathcal{L}(L^1(\Omega))$ *be positive such that* $\|S\|_{\mathcal{L}(L^1(\Omega))}, \|S\|_{\mathcal{L}(L^\infty(\Omega))} \leq M$. *Then there exists a unique family of consistent operators* $S_p \in \mathcal{L}(L^p(\Omega))$, $1 \leq p \leq \infty$, *such that*

1. $S = S_1$
2. $S_\infty$ *is an adjoint operator.*

To say that the family is **consistent** means

$$S_p f = S_q f \qquad \text{for all } f \in L^p(\Omega) \cap L^q(\Omega) \text{ and all } 1 \leq p, q \leq \infty.$$

To say that $S_\infty$ is an **adjoint operator** means that there exists an operator $T \in \mathcal{L}(L^1(\Omega))$ such that $S_\infty = T'$.

*Proof.*

(a) We have to show that there exists a unique operator $T \in \mathcal{L}(L^1(\Omega))$ such that $T'f = Sf$ for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$. Then we let $S_\infty := T'$. Uniqueness of $T$ is easy to see. In order to prove existence we note that the hypothesis implies that $S(L^1(\Omega) \cap L^\infty(\Omega)) \subset L^1(\Omega) \cap L^\infty(\Omega)$ and $\|Sf\|_{L^p(\Omega)} \leq M \|f\|_{L^p(\Omega)}$ for $p = 1, \infty$ and for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$. Observe that a function $g \in L^\infty(\Omega)$ is in $L^1(\Omega) \cap L^\infty(\Omega)$ and $\|g\|_{L^1(\Omega)} \leq M$ if and only if

$$\left| \int_\Omega g(x)\varphi(x)\, dx \right| \leq M \|\varphi\|_{L^\infty(\Omega)}$$

for all $\varphi \in L^1(\Omega) \cap L^\infty(\Omega)$. This shows that the adjoint $S' \in \mathcal{L}(L^\infty(\Omega))$ of $S$ leaves $L^1(\Omega) \cap L^\infty(\Omega)$ invariant and $\|S'f\|_{L^1(\Omega)} \leq M \|f\|_{L^1(\Omega)}$ for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$. By density of $L^1(\Omega) \cap L^\infty(\Omega)$ in $L^1(\Omega)$, there exists a unique operator $T \in \mathcal{L}(L^1(\Omega))$ such that $Tf = S'f$ for all $f \in L^1(\Omega) \cap L^\infty(\Omega)$. This implies that $T'g = Sg$ for all $g \in L^1(\Omega) \cap L^\infty(\Omega)$.

(b) By (a) there exists a linear, positive extension $\widetilde{S} : L^1(\Omega) + L^\infty(\Omega) \to L^1(\Omega) + L^\infty(\Omega)$ of $S$ such that $\|\widetilde{S}f\|_{L^\infty(\Omega)} \leq M \|f\|_{L^\infty(\Omega)}$ for all $f \in L^\infty(\Omega)$. In particular, $\widetilde{S}1 \leq M$.

(c) It follows from Lemma 4.3.2 that

$$S(f) \leq (\widetilde{S}f^p)^{\frac{1}{p}} (\widetilde{S}1)^{\frac{1}{p'}} \leq (\widetilde{S}f^p)^{\frac{1}{p}} M^{\frac{1}{p'}}$$

whenever $0 \leq f \in L^1(\Omega) \cap L^\infty(\Omega)$.

(d) By assumption $\|S\|_{\mathcal{L}(L^1(\Omega))} \leq M$. Let $0 \leq f \in L^1(\Omega) \cap L^\infty(\Omega)$. Then by (c)

$$\|Sf\|_{L^p(\Omega)} = \left(\int_\Omega (Sf)^p\right)^{1/p} \leq \left(\int_\Omega \tilde{S} f^p\right)^{1/p} M^{\frac{1}{p'}} \leq M^{\frac{1}{p}} \|f\|_{L^p} M^{\frac{1}{p'}} = M\|f\|_p$$

where $1 < p < \infty$. Since $L^1(\Omega) \cap L^\infty(\Omega)$ is dense in $L^p(\Omega)$, the operator $S|_{L^1(\Omega)\cap L^\infty(\Omega)}$ has a continuous extension $S_p \in \mathcal{L}(L^p(\Omega))$.

□

We add the interpolation inequality whose proof is also based on Lemma 4.3.2.

**Theorem 4.3.4** (Interpolation Inequality)**.** *Let $S : L^1(\Omega) \cap L^\infty(\Omega) \to L^1(\Omega) + L^\infty(\Omega)$ be positive and linear. Let*

$$1 < p, q < \infty, \ 0 < \theta < 1, \ \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

*Then*

$$\|S\|_{\mathcal{L}(L^r(\Omega))} \leq \|S\|^{1-\theta}_{\mathcal{L}(L^p(\Omega))} \cdot \|S\|^{\theta}_{\mathcal{L}(L^q(\Omega))} . \tag{4.13}$$

*Proof.* Since $|Sf| \leq S|f|$, we may assume that $f \geq 0$. Let $s = \frac{p}{(1-\theta)r}, s' = \frac{q}{\theta r}$. Then $1 < s < \infty$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Since $f = f^{r(1-\theta)/p} \cdot f^{r\theta/q}$ it follows from Lemma 4.3.2 that

$$Sf \leq (Sf^{r/p})^{1-\theta}(Sf^{r/q})^{\theta}.$$

Hence by Hölder's inequality,

$$\begin{aligned}
\int_\Omega (Sf)^r &\leq \left(\int_\Omega (Sf^{r/p})^{(1-\theta)rs}\right)^{1/s} \left(\int_\Omega (Sf^{r/q})^{\theta r s'}\right)^{1/s'} \\
&= \left\|Sf^{r/p}\right\|^{p/s}_{L^p(\Omega)} \cdot \left\|Sf^{r/q}\right\|^{q/s'}_{L^q(\Omega)} \\
&\leq \|S\|^{p/s}_{\mathcal{L}(L^p(\Omega))} \left\|f^{r/p}\right\|^{p/s}_{L^p(\Omega)} \cdot \|S\|^{q/s'}_{\mathcal{L}(L^q(\Omega))} \left\|f^{r/q}\right\|^{q/s'}_{L^q(\Omega)} \\
&= \|S\|^{(1-\theta)r}_{\mathcal{L}(L^p(\Omega))} \|f\|^{r/s}_{L^r(\Omega)} \cdot \|S\|^{\theta r}_{\mathcal{L}(L^q(\Omega))} \|f\|^{r/s'}_{L^r(\Omega)} .
\end{aligned}$$

Hence

$$\|Sf\|_{L^r(\Omega)} \leq \|S\|^{1-\theta}_{\mathcal{L}(L^p(\Omega))} \|S\|^{\theta}_{\mathcal{L}(L^q(\Omega))} \|f\|_{L^r(\Omega)} .$$

□

**Remark 4.3.5** (Riesz-Thorin Theorem)**.** *Theorem 4.3.4 remains true if the assumption that S be positive is omitted. This is precisely what the Riesz-Thorin Interpolation Theorem says.*

## 4.4 Extrapolation of semigroups

In this section we investigate how a submarkovian semigroup defined on $L^2(\Omega)$ can be extrapolated to $L^p(\Omega)$ and how properties of the given semigroup are inherited by the extended semigroup. We restrict ourselves to positive semigroups since most of our examples will be positive. In the comments we state more general results.

Let $T$ be a positive $C_0$-semigroup on $L^2(\Omega)$ with generator $A$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space. Assume that for $0 \le t \le 1$,

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega))} \le M \quad \text{and} \quad \|T(t)\|_{\mathcal{L}(L^\infty(\Omega))} \le M. \tag{4.14}$$

Then by (2.14) there exists $\omega \in \mathbb{R}$ such that

$$\|T(t)\|_{\mathcal{L}(L^p(\Omega))} \le Me^{\omega t} \quad (t \ge 0)$$

for $p = 1, \infty$. By Theorem 4.3.3 there exists a consistent family of operators $T_p(t) \in \mathcal{L}(L^p(\Omega))$ such that $\|T_p(t)\|_{\mathcal{L}(L^p(\Omega))} \le Me^{\omega t}$, where $T_\infty(t)$ is $\sigma(L^\infty(\Omega), L^1(\Omega))$-continuous, and $T_2(t) = T(t)$, the given semigroup. It is obvious from consistency that

$$T_p(t)T_p(s) = T_p(t+s) \quad (t, s \ge 0)$$

for all $1 \le p \le \infty$.

**Theorem 4.4.1.** *The function $T_p$ is a $C_0$-semigroup for $1 \le p < \infty$, i.e., $T_p$ is strongly continuous on $\mathbb{R}_+$.*

*Proof.*

1. Let $1 < p < 2$ and $0 < \theta < 1$ be such that $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$. Then by Hölder's inequality

$$\|(T_p(t) - T_p(t_0))f\|_{L^p(\Omega)} \le \|(T_2(t) - T_2(t_0))f\|^{\theta}_{L^2(\Omega)} \cdot \|(T_1(t) - T_1(t_0))f\|^{1-\theta}_{L^1(\Omega)}.$$

   Since $T_2(\cdot)f$ is continuous, also $T_p(\cdot)f$ is continuous.
   If $2 < p < \infty$ we write $1/p = (1-\theta)/\infty + \theta/2$ and argue similarly.

2. $p = 1$. The proof is given in several steps.

   (a) $T_1(t)f \wedge f \to f$ as $t \to 0+$ for each $0 \le f \in L^1(\Omega)$. In fact, consider first $0 \le f \in L^1(\Omega) \cap L^2(\Omega)$. Let $t_n \downarrow 0$. Considering a subsequence, if necessary, such that $T_2(t_n)f \to f$ a.e. . Hence $T_1(t_n)f \wedge f \to f$ in $L^1(\Omega)$ by the Dominated Convergence Theorem. Since $L^1_+(\Omega) \cap L^2(\Omega)$ is dense in $L^1_+(\Omega)$ the claim follows.

(b) Let $f \in L^1(\Omega)$. We show that the set $\{T_1(t)f : 0 < t \leq 1\}$ is weakly compact in $L^1(\Omega)$. We may assume that $f \geq 0$. Since $L^1_+(\Omega) \cap L^2(\Omega)$ is dense in $L^2_+(\Omega)$ we may also assume that $f \in L^1_+(\Omega) \cap L^2(\Omega)$. Recall that for $u \in L^1_+(\Omega)$ the order interval $[0, u] := \{g \in L^1(\Omega) : 0 \leq g \leq u\}$ is weakly compact [Sch74].

For $\quad f_n := n \int\limits_0^{1/n} (T_2(s)f \wedge f)ds \quad$ we get that $\quad T_1(t)f_n \leq n \int\limits_0^2 T_2(s)f \, ds =: u.$

Since $\|T_2(s)f\|_{L^1(\Omega)} \leq M \|f\|_{L^1(\Omega)}$ it follows that

$$|< u, \varphi >| \leq n2M \|f\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)}$$

for all $\varphi \in L^2(\Omega) \cap L^\infty(\Omega)$. Hence $u \in L^1(\Omega)$ (observe that $u$ is the limit in $L^2(\Omega)$ of Riemann sums). Then $\{T_1(t)f_n : 0 < t \leq 1\}$ is contained in an order interval and hence is weakly compact. Since $f_n \to f$ in $L^1(\Omega)$ by (2a), it follows that also $\{T_1(t)f : 0 < t \leq 1\}$ is weakly compact.

(c) Since $L^1(\Omega)$ is separable, there is a dense sequence $\{f_i : i \in \mathbb{N}\}$ in $L^1(\Omega)$. Let $t_n \downarrow 0$. By (2b) and by a diagonal sequence argument, we find a subsequence such that $T(t_{n_l})f_i$ converges weakly as $l \to \infty$ for all $i \in \mathbb{N}$. Hence $Pf := \lim_{l\to\infty} T(t_{n_l})f$ converges weakly for all $f \in L^1(\Omega)$. Then $P \in \mathcal{L}(L^1(\Omega))$ and $Pf = f$ for all $f \in L^1(\Omega) \cap L^2(\Omega)$. Consequently, $Pf = f$ for all $f \in L^1(\Omega)$. We have shown that $T(t)f$ converges weakly to $f$ as $t \downarrow 0$. By a general result on semigroups [Dav80, Proposition 1.2.3], this implies that $T$ is a $C_0$-semigroup.

□

Next we want to discuss the semigroup $T_\infty$. We will introduce the notion of a dual semigroup. Let $T$ be a $C_0$-semigroup on a Banach space $X$, then $(T'(t))_{t\geq 0}$ is a family of operators on $X'$ satisfying

$$T'(t + s) = T'(t)T'(s) \quad s, t \geq 0.$$

But $T'(\cdot)x'$ is merely continuous for the $\sigma(X', X)$-topology. We call $T'$ a **dual semigroup** and the adjoint $A'$ of $A$ the **generator** of $T'$.
If $X$ is reflexive, then $T'$ is a $C_0$-semigroup (see [ABHN01, Corollary 3.3.9]). Coming back to the semigroups $T_p, 1 \leq p \leq \infty$, introduced before, it is easy to see that $T_\infty$ is a dual semigroup. In fact, $S = T'$ is a $C_0$-semigroup on $L^2(\Omega)$, since $L^2$ is reflexive. Thus $S_1$ is a $C_0$-semigroup by Theorem 4.4.1. It is clear that $T_\infty(t) = S_1(t)'$.

We call the family $(T_p(t))_{t\geq 0}, 1 \leq p \leq \infty$, the **extrapolation semigroups** of $T$. Next we investigate how some properties are inherited by $T_p$ from $T_2$.

**Theorem 4.4.2.** *Assume that $T(t)$ is compact for $t > 0$. Then also $T_p(t)$ is compact for $1 < p < \infty, t > 0$.*

This follows from the following result applied to $T(t)$ and $T'(t)$.

**Proposition 4.4.3.** *Let $1 \le p_0 < p_1 \le \infty$ and let $A_p \in \mathcal{L}(L^p(\Omega))$, $p_0 \le p < p_1$, be a* **consistent** *family of operators, i.e.,*

$$A_p f = A_q f \quad \text{for} \quad f \in L^p(\Omega) \cap L^q(\Omega), \;\; p_0 \le p, q < p_1 \; .$$

*If $A_{p_0}$ is compact, then also $A_p$ is compact for all $p \in [p_0, p_1)$.*

*Proof.* Denote by $I$ the set of all sets of finite disjoint subsets $E_1, \ldots, E_m$ with $0 < |E_j| < \infty$ ordered by inclusion. For $i \in I$,

$$P_i f := \sum_{j=1}^{m} |E_j|^{-1} \int\limits_{E_j} f(x) dx \; 1_{E_j}$$

defines a finite rank projection on $L^q(\Omega)$ with norm $\|P_i\|_{\mathcal{L}(L^q(\Omega))} \le 1$ for all $1 \le q < \infty$. Moreover $\lim_i P_i f = f$ in $L^q(\Omega)$ for all $f \in L^q(\Omega)$. This limit is uniform on compact subsets of $L^q(\Omega)$. Since $A_{p_0}$ is compact, it follows that

$$\lim_i \|A_{p_0} - P_i A_{p_0}\|_{\mathcal{L}(L^{p_0}(\Omega))} = 0.$$

Now let $p_0 < p < p_1$. Write

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \quad \text{with} \quad 0 < \theta < 1.$$

Then

$$\begin{aligned} \|A_p - P_i A_p\|_{\mathcal{L}(L^p(\Omega))} &\le \|A_{p_0} - P_i A_{p_0}\|^{\theta}_{\mathcal{L}(L^{p_0}(\Omega))} \, \|A_{p_1} - P_i A_{p_1}\|^{1-\theta}_{\mathcal{L}(L^{p_1}(\Omega))} \\ &\le \|A_{p_0} - P_i A_{p_0}\|^{\theta}_{\mathcal{L}(L^{p_0}(\Omega))} \left(2 \, \|A_{p_1}\|_{\mathcal{L}(L^{p_1}(\Omega))}\right)^{1-\theta} \to 0 \end{aligned}$$

with respect to $i \in I$. This follows from the Interpolation Inequality (Theorem 4.3.4). Since $P_i A_p$ is compact, it follows that $A_p$ is also compact. ☐

Next we consider holomorphy. We assume that $M = 1$ in (4.14), i.e., that $T$ and $T'$ are submarkovian.

**Theorem 4.4.4** (extrapolation of holomorphy)**.** *Assume that $T$ and $T'$ are submarkovian. If $T$ is holomorphic, then also $T_p$ is holomorphic for $1 < p < \infty$.*

*Proof.* It follows from Theorem 2.5.6 or Exercise 2.6.3 that $\overline{\lim}_{t \downarrow 0} \|T(t) - I\|_{\mathcal{L}(L^2(\Omega))} < 2$. Let $1 < p < 2$. Then $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ for some $0 < \theta < 1$. By the Interpolation Inequality (Theorem 4.3.4) it follows that

$$\overline{\lim_{t \downarrow 0}} \, \|T_p(t) - I\|_{\mathcal{L}(L^p(\Omega))} \le \overline{\lim_{t \downarrow 0}} \left( \|T_1(t) - I\|^{\theta}_{\mathcal{L}(L^1(\Omega))} \, \|T(t) - I\|^{1-\theta}_{\mathcal{L}(L^2(\Omega))} \right) < 2^{\theta} 2^{1-\theta} = 2.$$

Now it follows from Theorem 2.5.5 that $T_p$ is holomorphic. ☐

## 4.5 Exercises

**Exercise 4.5.1.** *Give a proof of* (4.9).

The next exercise gives a measure theoretic argument needed in the proof of the Dunford–Pettis Theorem. Let $\Omega \subset \mathbb{R}^n$ be open. If $A$ is a Borel set in $\Omega \times \Omega$ then it is well-known that $A_1(y) = \{x \in \Omega : (x, y) \in A\}$ is a Borel set for all $y \in \Omega$.

**Exercise 4.5.2.**

1. *Let $A \subset \Omega \times \Omega$ be a Borel set. Show that the following assertions are equivalent:*
   (a) *$A$ has a Lebesgue measure $0$;*
   (b) *there is a Borel null set $N$ in $\Omega$ such that for each $y \in \Omega \setminus N$ the set $A_1(y)$ has measure zero.*

   **Hint:** *Use Fubini's theorem.*
2. *Convince yourself that assertion 1 can be reformulated in the following way:*
   *Let $P(x, y)$ be an assertion for each $(x, y) \in \Omega\times\Omega$. Then $P(x, y)$ is true for almost all $(x, y) \in \Omega\times\Omega$ if and only if for almost all $y \in \Omega$, $P(x, y)$ holds $x$ -a.e..*
3. *Let $f_i, g_i \in L^1(\Omega)$ such that $u(x, y) = \sum_{i=1}^{m} f_i(x)g_i(y) = 0$ a.e. and let $B \in \mathcal{L}(L^1(\Omega))$. Show that*
$$\sum_{i=1}^{m} \int_\Omega (Bf_i)(y)g_i(y)dy = 0.$$

**Exercise 4.5.3** (extrapolation of strong continuity to $L^1(\Omega)$)**.**
*Let $\Omega \subset \mathbb{R}^n$ be open, $|\Omega| < \infty$ and let $T_2$ be a $C_0$-semigroup on $L^2(\Omega)$ such that*
$$\sup_{0<t\le 1} \|T_2(t)\|_{\mathcal{L}(L^1(\Omega))} < \infty.$$
*Denote by $T_1(t) \in \mathcal{L}(L^1(\Omega))$ the continuous extension of $T_2(t)$.*
*Show that $\lim_{t\downarrow 0} T_1(t)f = f$ in $L^1(\Omega)$ for all $f \in L^1(\Omega)$.* **Hint:** *Use that $L^2(\Omega) \hookrightarrow L^1(\Omega)$.*

In the following exercises $T$ is a $C_0$-semigroup on $L^2(\Omega)$ such that $T$ and $T'$ are submarkovian. $T_p$ is the extrapolation semigroup on $L^p(\Omega), 1 \le p < \infty$. The solution depends on the interpolation inequality (Theorem 4.3.4) in each case.

**Exercise 4.5.4** (extrapolation of exponential stability)**.** *Assume that $T$ is* exponentially stable*, that is,*
$$\|T(t)\|_{\mathcal{L}(L^2)} \le Me^{-\varepsilon t}, t \ge 0,$$
*for some $\varepsilon > 0$, $M \ge 1$. Show that $T_p$ is exponentially stable for $1 < p < \infty$.*

**Exercise 4.5.5** (extrapolation of strong stability)**.**

1. *Assume that $T$ is* strongly stable*, that is, $\lim_{t\to\infty} T(t)f = 0$ in $L^2(\Omega)$ for all $f \in L^2(\Omega)$. Show that $T_p$ is* strongly stable *($1 < p < \infty$).*
2. *Show that the Gaussian semigroup $G_2$ on $L^2(\mathbb{R}^n)$ is strongly stable.*
   **Hint:** *Use that $\mathcal{F}G_2(t)\mathcal{F}^{-1} = S(t)$ with $(S(t)f)(x) = e^{-t|x|^2}f(x)$ where $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denotes the Fourier-Plancherel transform (see [ISEM99/00]).*

*3. Let $T(t) = e^{t\Delta_\Omega^D}$ where $\Omega \subset \mathbb{R}^n$ is an open set. Show that $T_p$ is strongly stable* $(1 < p < \infty)$.

*4. Show that the Gaussian semigroup $G_1$ is not strongly stable in $L^1(\mathbb{R}^n)$.*
**Hint:** *Show that $G_1'(t)1_{\mathbb{R}^n} = 1_{\mathbb{R}^n}$ $(t \geq 0)$ by using Theorems 3.1.7 and 4.3.3.*

**Exercise 4.5.6** (extrapolation of norm continuity at 0)**.**
*Assume that $T$ has a bounded generator. Show that $T_p$ has a bounded generator* $(1 < p < \infty)$.

# 4.6 Comments

Several of the results presented in this lecture were not formulated in optimal generality. The reason is mainly that a simple (frequently elegant) proof could be given in a special case or that the special cases we consider here suffice for the applications we have in mind. Here we give comments on the diverse results and explain how they can be generalized.

## 4.6.1 The Dunford–Pettis Theorem

The proof we give here is taken from Arendt-Bukhvalov [AB94]. It seems that the theorem had first been obtained by Kantorovich and Vulikh, see [AB94] for further references. But we followed the nomenclature which is most common in the literature.

## 4.6.2 Compactness from Dunford-Pettis

Here is a theorem on compactness which is of interest in the context of kernel operators.

**Theorem 4.6.1.** *Let $X$ be a reflexive Banach space and $T \in \mathcal{L}(X, L^q(\Omega))$, $1 \leq q < \infty$, where $\Omega \subset \mathbb{R}^n$ is open and $|\Omega| < \infty$. Assume that $T(X) \subset L^\infty(\Omega)$. Then $T$ is compact.*

*Proof.* By the Gelfand-Naimark Theorem we may identify $L^\infty(\Omega)$ with a space $C(K)$ and $L^q(\Omega)$ with $L^q(K, \mu)$ where $K$ is compact and $\mu$ a Borel measure on $K$. Denote by $j : C(K) \to L^q(K, \mu)$ the injection and $S \in \mathcal{L}(X, C(K))$ such that $T = j \circ S$. Let $x_n \in X$, $\|x_n\| \leq 1$. We have to show that $(Tx_n)$ has a convergent subsequence. Since $X$ is reflexive we may assume that $x_n \rightharpoonup x$ as $n \to \infty$, taking a subsequence otherwise. Let $\omega \in K$ and $\delta_\omega$ the corresponding Dirac measure. Then

$$(Sx_n)(\omega) = \langle \delta_\omega, Sx_n \rangle = \langle S'\delta_\omega, x_n \rangle \to \langle S'\delta_\omega, x \rangle = (Sx)(\omega).$$

Since $S$ is bounded,

$$\|Sx_n\|_{C(K)} \leq \|S\| \sup_{n \in \mathbb{N}} \|x_n\|_X.$$

Now it follows from the Dominated Convergence Theorem that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} j(Sx_n) = j(Sx) = Tx \quad \text{in } L^q(K, \mu).$$

□

The result is no longer valid if $X$ is not reflexive (take $X = L^\infty$ and $T$ the injection of $L^\infty$ in $L^q$).

In the context of the Dunford-Pettis Theorem we obtain the following

**Corollary 4.6.2.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $|\Omega| < \infty$, $1 < p < \infty$ and let $T \in \mathcal{L}(L^p(\Omega))$ be such that $TL^p(\Omega) \subset L^\infty(\Omega)$. Then $T$ is compact.*

### 4.6.3 Interpolation Inequality

The beautiful proofs of Theorem 4.3.3 and Theorem 4.3.4 are taken from Haase [Haa04]. They are completely elementary and depend on positivity, whereas the Riesz-Thorin Theorem is proved by complex methods, see e.g. [RS72].

### 4.6.4 Extrapolation of the $C_0$-property

Let $T$ be a $C_0$-semigroup on $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. Assume that $\sup_{0<t\leq 1} \|T(t)\|_{\mathcal{L}(L^1(\Omega))} < \infty$. Then there exist operators $T_1(t) \in \mathcal{L}(L^1(\Omega))$ such that $T_1(t)f = T(t)f$ for all $f \in L^1(\Omega) \cap L^2(\Omega)$. It is obvious that $T_1(t+s) = T_1(t)T_1(s)$ for all $t, s \geq 0$. Moreover, $T_1$ is strongly continuous on $(0, \infty)$. But it seems to be open whether $T_1$ is a $C_0$-semigroup. It is if one of the following conditions is satisfied.

1. $|\Omega| < \infty$ (see Exercise 4.5.3),
2. $\|T(t)\|_{\mathcal{L}(L^1)} \leq 1$ (see [Dav89, Theorem 1.4.1]),
3. $T(t) \geq 0$ (see Theorem 4.4.1 which is due to Voigt [Voi92]).

We refer to Voigt [Voi92] for further results and a general discussion.

### 4.6.5 $p$-independence of the spectrum

Let $T_p$ be a consistent family of $C_0$-semigroup on $L^p(\Omega), 1 < p < \infty$, with generator $A_p$. If $A_2$ has compact resolvent then also $A_p$ has compact resolvent. This follows from Proposition 4.4.3. In that case one also has that $\sigma(A_p) = \sigma(A_2)$, $1 < p < \infty$ and each eigenfunction of $A_2$ lies in $\bigcap_{1<p<\infty} L^p(\Omega)$, cf. [Dav89, § 4.3].

### 4.6.6 Extrapolation of holomorphy

A more general result than Theorem 4.4.3 is valid. Let $T_p$ be a consistent family of $C_0$-semigroups on $L^p(\Omega)$, $1 < p < \infty$. If $T_2$ is holomorphic, then also $T_p$ is holomorphic, $1 < p < \infty$. This follows from Stein's Interpolation Theorem, see [Dav89].

### 4.6.7 Heritage List

There is more on extrapolating properties in the survey article [Are04, 7.2.2].

### 4.6.8 Automatic Continuity

Each positive linear mapping $T : X \to Y$ where $X, Y$ are Banach lattices is continuous. For example $X$ and $Y$ may be spaces as $L^p$ $(1 \leq p \leq \infty)$, $C_0(\Omega)$ or $C(\overline{\Omega})$. In fact, since $|Tf| \leq T|f|$ and $\| \, |f| \, \| = \|f\|$, it suffices to prove that $\|Tf\| \leq c\|f\|$ for some $c > 0$ and all $f \in X_+$. If such $c$ does not exist, then we can find $f_n \in X_+$ such that $\|f_n\| \leq 2^{-n}$ but $\|Tf_n\| \geq n$. Let

$$f := \sum_{n=1}^{\infty} f_n.$$

Then $f \in X_+$ and so $0 \leq Tf_n \leq Tf$, hence $n \leq \|Tf_n\| \leq \|Tf\|$ for all $n \in \mathbb{N}$. This is a contradiction.

# Lecture 5

# Continuity of the Kernels

In this lecture, we give a precise condition for the kernel of the semigroup generated by the Dirichlet Laplacian to be continuous up to the boundary. We start by investigating when the Dirichlet Laplacian generates a $C_0$-semigroup on the space $C_0(\Omega)$. This lecture is divided into three parts:

5.1 The Gaussian semigroup revisited

5.2 The Dirichlet Laplacian on $C_0(\Omega)$

5.3 Continuity of the kernel at the boundary

## 5.1 The Gaussian semigroup revisited

In this brief section, we want to describe the Gaussian semigroup on the Banach space

$$C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0\}.$$

We recall **Young's Inequality**. Let $k \in L^1(\mathbb{R}^n)$. For $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, we define

$$(k * f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy.$$

Then $k * f \in L^p(\mathbb{R}^n)$ and $\|k * f\|_{L^p(\mathbb{R}^n)} \le \|k\|_{L^1}\|f\|_{L^p(\mathbb{R}^n)}$. If $f \in C_0(\mathbb{R}^n)$, then also $k * f \in C_0(\mathbb{R}^n)$.

Now consider the **Gaussian kernel**

$$g_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}.$$

Then $g_t \in L^1(\mathbb{R}^n)$ and $\|g\|_{L^1} = 1$. The Gaussian semigroup is given by

$$G_p(t)f := g_t * f \qquad\qquad (f \in L^p(\mathbb{R}^n))$$

for $1 \le p \le \infty$. We also define

$$G_0(t)f := g_t * f \qquad\qquad (f \in C_0(\mathbb{R}^n)).$$

Then $G_p$ is a $C_0$-semigroup on $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and on $C_0(\mathbb{R}^n)$ for $p = 0$. We have that $G_1(t)' = G_\infty(t)$, so $G_\infty$ is a dual semigroup.

Defining $g_z(x) = (4\pi z)^{-n/2} e^{-|x|^2/4z}$ for $\operatorname{Re} z > 0$, also $g_z \in L^1(\mathbb{R}^n)$, and $\|g_z\|_{L^1(\mathbb{R}^n)} = (\operatorname{Re} z)^{-n}$. The function $z \mapsto g_z$ from $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ into $L^1(\mathbb{R}^n)$ is holomorphic. Thus we obtain a holomorphic extension of $G_p$ to $\mathbb{C}_+$ given by $G_p(z)f = f * g_z$, where $1 \le p \le \infty$ or $p = 0$. Thus $G_p$ is a holomorphic $C_0$-semigroup on $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and on $C_0(\mathbb{R}^n)$ for $p = 0$. Next we describe the generator of $G_p$.

**Proposition 5.1.1.** *The generator of $G_p$ is the* **distributional Laplacian** $\Delta_p$, *i.e.,*

$$D(\Delta_p) = \{f \in L^p(\mathbb{R}^n) : \Delta f \in L^p(\mathbb{R}^n)\};$$
$$D(\Delta_0) = \{f \in C_0(\mathbb{R}^n) : \Delta f \in C_0(\mathbb{R}^n)\}$$

*and* $\Delta_p f = \Delta f$ *in* $\mathcal{D}(\mathbb{R}^n)'$.

We make this more precise. In the case $p = 0$, for example, this means the following. Let $f, g \in C_0(\mathbb{R}^n)$. Then $f \in D(\Delta_0)$ and $\Delta f = g$ if and only if

$$\int_{\mathbb{R}^n} f\Delta\varphi = \int_{\mathbb{R}^n} g\varphi \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We mention explicitly that $D(\Delta_0) \not\subset C^2(\mathbb{R}^n)$ if $n \ge 2$.

*Proof of Proposition 5.1.1.* We give the proof in the case of $p = 0$. Since $G_0$ is holomorphic, we have $G_0(t)f \in D(\Delta_0)$ for all $t > 0$, $f \in C_0(\mathbb{R}^n)$. Let $f, g \in C_0(\mathbb{R}^n)$.

a) Assume that $f \in D(\Delta_0)$ and $\Delta_0 f = g$. Then $\lim_{t\to 0} \Delta G_0(t)f = g$ in $C_0(\mathbb{R}^n)$. Thus

$$\int_{\mathbb{R}^n} f\Delta\varphi = \lim_{t\to 0}\int_{\mathbb{R}^n} (G_0(t)f)\Delta\varphi = \lim_{t\to 0}\int_{\mathbb{R}^n} (\Delta G_0(t)f)\varphi = \int_{\mathbb{R}^n} g\varphi.$$

Hence $\Delta f = g$ in $\mathcal{D}(\mathbb{R}^n)'$.

b) Conversely, assume that $\Delta f = g$ in $\mathcal{D}(\mathbb{R}^n)'$. Then for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $t > 0$, by Fubini's Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} \Delta_0(G_0(t)f)\varphi &= \int_{\mathbb{R}^n} (G_0(t)f)\Delta\varphi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g_t(y)f(x-y)dy\Delta\varphi(x)dx = \int_{\mathbb{R}^n} g_t(y)\int_{\mathbb{R}^n} f(x-y)\Delta\varphi(x)dxdy \\
&= \int_{\mathbb{R}^n} g_t(y)\int_{\mathbb{R}^n} f(x)\Delta\varphi(x+y)dxdy = \int_{\mathbb{R}^n} g_t(y)\int_{\mathbb{R}^n} g(x)\varphi(x+y)dxdy \\
&= \int_{\mathbb{R}^n} g_t(y)\int_{\mathbb{R}^n} g(x-y)\varphi(x)dxdy = \int_{\mathbb{R}^n} (G_t g)(x)\varphi(x)dx.
\end{aligned}$$

Hence $\Delta_0 G_t f = G_t g$ $(t > 0)$. Since $\Delta_0$ is closed and $G_t g \to g$, $G_t f \to f$ in $C_0(\mathbb{R}^n)$ as $t \downarrow 0$, it follows that $f \in D(\Delta_0)$ and $\Delta_0 f = g$.

□

## 5.2 The Dirichlet Laplacian on $C_0(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with boundary $\Gamma$. A function $h \in C(\Omega)$ is called **harmonic** if

$$\Delta h = 0 \text{ in } \mathcal{D}(\Omega)'. \tag{5.1}$$

This means that

$$\int_\Omega h \Delta v dx = 0 \text{ for all } v \in \mathcal{D}(\Omega)',$$

and it implies that $h \in C^\infty(\Omega)$ and $\Delta h = 0$ in the classical sense.

**Definition 5.2.1.** *The set $\Omega$ is called* **Dirichlet regular** *if for all $\varphi \in C(\Gamma)$ there exists a solution of the* **Dirichlet Problem**

$$(D_\varphi) \qquad \begin{cases} h \in C(\overline{\Omega}), \\ h_{|\Gamma} = \varphi, \\ \Delta h = 0 \quad \text{in } D(\Omega)'. \end{cases}$$

Thus we look for a harmonic function in $\Omega$, which is continuous up to the boundary $\Gamma$ and takes the prescribed value $\varphi$ on $\Gamma$.

The **Elliptic Maximum Principle** says that

$$\max_{\overline{\Omega}} h = \max_\Gamma h = \max_\Gamma \varphi \tag{5.2}$$

for each solution $h$ of $(D_\varphi)$ (see Exercise 5.4.4). This shows in particular that $h \leq 0$ on $\overline{\Omega}$ if $\varphi \leq 0$ and hence $h \equiv 0$ if $\varphi \equiv 0$. Thus there exists at most one solution of $(D_\varphi)$.

Dirichlet regularity is a property of the boundary $\Gamma$ of $\Omega$. We give some further examples.

**Examples 5.2.2.** *a) If $n = 1$, then each bounded open set in $\mathbb{R}$ is Dirichlet regular.*

*b) Each simply connected bounded, open subset of $\mathbb{R}^2$ is Dirichlet regular.*

*c) If $\Omega \subset \mathbb{R}^n$ is bounded and open, and if $z \in \Omega$, then $\Omega \setminus \{z\}$ is not Dirichlet regular.*

*d) If $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary, then $\Omega$ is Dirichlet regular.*

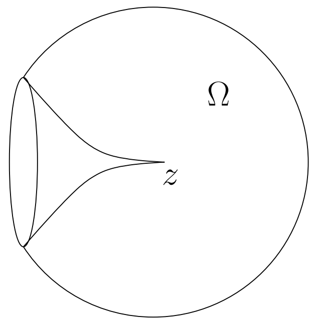


Figure 5.1: A cusp in $z \in \Gamma$.

*e) If $\Omega \subset \mathbb{R}^3$ has an entering cusp which is sufficiently sharp, then $\Omega$ is not Dirichlet regular.*

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We consider the Banach space

$$C_0(\Omega) := \{u \in C(\overline{\Omega}) : u_{|\Gamma} = 0\}$$

with the supremum norm

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|.$$

Define the realisation $\Delta_0^\Omega$ of the Laplacian on $C_0(\Omega)$ by

$$\begin{aligned} D(\Delta_0^\Omega) &:= \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}, \\ \Delta_0^\Omega u &= \Delta u, \end{aligned}$$

where $\Delta u$ is understood in the sense of $\mathcal{D}(\Omega)'$. Thus for $u, f \in C_0(\Omega)$, we have that $u \in D(\Delta_0^\Omega)$ and $\Delta_0^\Omega u = f$ if and only if

$$\int_\Omega u \Delta v dx = \int_\Omega f v dx \quad (v \in \mathcal{D}(\Omega)). \tag{5.3}$$

Our aim is to prove the following.

**Theorem 5.2.3.** *If $\Omega$ is Dirichlet regular, then $\Delta_0^\Omega$ generates a holomorphic $C_0$-semigroup on $C_0(\Omega)$.*

We need the following maximum principle for complex-valued functions.

**Proposition 5.2.4** (Maximum Principle)**.** *Let $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > 0$, $v \in C(\overline{\Omega})$ such that $\lambda v - \Delta v = 0$ in $\mathcal{D}(\Omega)'$. Then*

$$\max_{z \in \Gamma} |v(z)| = \max_{x \in \overline{\Omega}} |v(x)|.$$

*Proof.* Suppose that $m := \max_{z\in\Gamma} |v(z)| < \|v\|_{L^\infty(\Omega)}$. Let $\varepsilon = \|v\|_{L^\infty(\Omega)} - m$.

Let
$$v_k(x) = (\rho_k * v)(x) = \int_{\mathbb{R}^n} v(x-y)\rho_k(y)dy$$
where $(\rho_k)_{k\in\mathbb{N}}$ is a mollifier (see the proof of Proposition 3.2.8). Then $v_k \in C^\infty(\mathbb{R}^n)$, $\|v_k\|_{L^\infty(\mathbb{R}^n)} \le \|v\|_{L^\infty(\Omega)}$ for all $k \in \mathbb{N}$ and $v_k \to v$ as $k \to \infty$ uniformly on compact subsets of $\Omega$.

The set $K := \{x \in \Omega : |v(x)| \ge m + 2\varepsilon/3\}$ is compact and $U := \{x \in \overline{\Omega} : |v(x)| < m + \varepsilon/3\}$ has positive distance $0 < \delta = \operatorname{dist}(U, K)$ from $K$ (since $U$ is relatively compact and $\overline{U} \cap K = \emptyset$).

Let $k_0 > 1/\delta$. Then for $k \ge k_0$, $x \in U$, $|v(x-y)| \le m + 2\varepsilon/3$ whenever $|y| < 1/k$. Hence
$$|v_k(x)| \le \int_{|y|<1/k} \rho_k(y)|v(x-y)| \le m + \frac{2\varepsilon}{3}.$$

Let $K_1 = \Omega \setminus U$. Then $K_1$ is compact. Thus $v_k$ converges to $v$ uniformly on $K_1$, hence $\|v_k\|_{L^\infty(K_1)} \to \|v\|_{L^\infty(K_1)} = \|v\|_{L^\infty(\Omega)}$. Moreover since $|v_k(x)| \le m + \frac{2\varepsilon}{3}$ for $x \notin K_1$, for sufficiently large $k$, there exist $x_k \in K_1$ such that $|v_k(x_k)| = \max_{x\in\mathbb{R}^n} |v_k(x)|$. Hence $f_k(x) = \operatorname{Re}[v_k(x)\overline{v_k(x_k)}]$ has a maximum at $x_k$. Consequently,

$$\operatorname{Re}[\Delta v_k(x_k)\overline{v}_k(x_k)] = \Delta f_k(x_k) \le 0. \tag{5.4}$$

Taking a subsequence if necessary, we may assume that $x_k \to x_0$ in $K$ as $k \to \infty$. Since $v_k \to v$ in $L^\infty(K)$, it follows that
$$|v(x_0)| = \lim_{k\to\infty} |v_k(x_k)| = \lim_{k\to\infty} \|v_k\|_{L^\infty(K_1)} = \|v\|_{L^\infty(K_1)} = \|v\|_{L^\infty(\Omega)}.$$

Note that $\Delta v_k = \Delta v * \rho_k \to \Delta u$ uniformly on $K_1$ as $k \to \infty$. It follows from (5.4) that
$$\operatorname{Re}[\Delta v(x_0)\overline{v(x_0)}] \le 0.$$
Hence
$$\operatorname{Re}\lambda|v(x_0)|^2 \le \operatorname{Re}\lambda|v(x_0)|^2 - \operatorname{Re}[\overline{v}(x_0)\Delta v(x_0)] = \operatorname{Re}[\overline{v(x_0)}(\lambda v(x_0) - \Delta v(x_0))] = 0.$$
But $|v(x_0)| \ge m + \varepsilon/3$, since $x_0 \in K$, which is a contradiction. □

Denote by $E_n$ the **Newtonian Potential**, i.e. $E_n : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by
$$E_n(x) := \begin{cases} \frac{|x|}{2} & \text{if } n = 1, \\ \frac{\log|x|}{2\pi} & \text{if } n = 2, \\ -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{if } n \ge 3, \end{cases}$$
where $\omega_n = |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^n$. Then $E_n \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $E_n, D_jE_n \in L^1_{\text{loc}}(\mathbb{R}^n)$, as it is easy to see.

**Proposition 5.2.5.** *Let $f \in C_c(\mathbb{R}^n)$, $v = E_n * f$. Then $v \in C^1(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathcal{D}(\mathbb{R}^n)'$. Here*

$$(E_n * f)(x) = \int_{\mathbb{R}^n} f(x-y)E_n(y)dy.$$

We refer to [DL88, Chapter II, Section 3] for this standard fact.

*Proof of Theorem 5.2.3.* We consider first the case when $\mathbb{K} = \mathbb{C}$. The space $C_0(\Omega)$ may be seen as a subspace of $C_0(\mathbb{R}^n)$ by extending functions by 0 outside of $\Omega$.

a) We show that $\Delta_0$ is invertible. Let $f \in C_0(\Omega)$, $v = E_n * f$. Then $v \in C(\overline{\Omega})$ and $\Delta v = f$ in $\mathcal{D}(\Omega)'$. Let $\varphi = v_{|\Gamma}$. Let $h \in C(\overline{\Omega})$ be the solution of $(D_\varphi)$ and $u = v - h$. Then $u_{|\Gamma} = 0$ and $\Delta u = \Delta v = f$ in $\mathcal{D}(\Omega)'$. We have shown that $\Delta_0$ is surjective. The Elliptic Maximum Principle (5.2) implies that $\Delta_0$ is injective. It is clear that $\Delta_0$ is closed. Thus $\Delta_0$ is invertible.

b) The operator $\Delta_0$ generates a holomorphic $C_0$-semigroup on $C_0(\mathbb{R}^n)$, which is bounded on a sector $\Sigma_\theta$, $0 < \theta < \frac{\pi}{2}$. Thus $\mathbb{C}_+ \subset \rho(\Delta_0)$ and $\|\lambda R(\lambda, \Delta_0)\| \le M$ for $\operatorname{Re}\lambda > 0$ (see e.g., [ABHN01, Corollary 3.7.12]).

Let $u \in D(\Delta_0^\Omega)$. We claim that

$$\|\lambda u\|_{L^\infty(\Omega)} \le 2M\|\lambda u - \Delta u\|_{L^\infty(\Omega)} \quad (\operatorname{Re}\lambda > 0). \tag{5.5}$$

In fact, let $f = \lambda u - \Delta u \in C_0(\Omega) \subset C_0(\mathbb{R}^n)$, $v = R(\lambda, \Delta_0)f$. Then $\|\lambda v\|_{L^\infty(\mathbb{R}^n)} \le M\|f\|_{L^\infty(\mathbb{R}^n)}$ and

$$\lambda(u-v) - \Delta(u-v) = 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

It follows from the Maximum Principle, Proposition 5.2.4, that

$$\|u-v\|_{L^\infty(\Omega)} = \max_\Gamma |u-v| = \max_\Gamma |v| \le \frac{M}{|\lambda|}\|f\|_{L^\infty(\mathbb{R}^n)}.$$

Hence

$$\|\lambda u\|_{L^\infty(\Omega)} \le \|\lambda(u-v)\|_{L^\infty(\Omega)} + \|\lambda v\|_{L^\infty(\Omega)} \le M\|f\|_{L^\infty(\Omega)} + M\|f\|_{L^\infty(\Omega)}.$$

c) Since $\rho(\Delta_0^\Omega)$ is open, it follows from a) that $O = \rho(\Delta_0^\Omega) \cap \mathbb{C}_+$ is open and non-empty. From b), we deduce that

$$\|\lambda R(\lambda, \Delta_0^\Omega)\| \le 2M \tag{5.6}$$

for all $\lambda \in O$. It follows from Proposition 1.2.2 that for $\lambda_0 \in O$,

$$\operatorname{dist}(\lambda_0, \sigma(\Delta_0^\Omega)) \ge \|R(\lambda_0, \Delta_0^\Omega)\|^{-1} \ge \frac{|\lambda_0|}{2M}.$$

This implies that $O$ is relatively closed in $\mathbb{C}_+$. Since $\mathbb{C}_+$ is connected, we conclude that $O = \mathbb{C}_+$. Since $\mathcal{D}(\Omega) \subset D(\Delta_0^\Omega)$, the operator $\Delta_0^\Omega$ is densely defined. Since $\mathbb{C}_+ \subset \rho(\Delta_0^\Omega)$ and (5.6) is valid for all $\lambda \in \mathbb{C}_+$, we deduce that $\Delta_0$ generates a holomorphic $C_0$-semigroup by Theorem 2.5.3.

Since the space of all real-valued functions in $C_0(\Omega)$ is invariant under $R(\lambda, \Delta_0^\Omega)$, it follows from Euler's Formula (2.15) that also $e^{t\Delta_0^\Omega}$ leaves the real space $C_0(\Omega)$ invariant. ◻

Finally, we prove that the semigroup $e^{t\Delta_0^\Omega}$ on $C_0(\Omega)$ is consistent with the semigroup generated by the Dirichlet Laplacian $e^{t\Delta_\Omega^D}$ on $L^2(\Omega)$ (see Theorem 3.1.4). We continue to assume that $\Omega$ is Dirichlet regular.

**Lemma 5.2.6.** *$D(\Delta_0^\Omega) \subset C^1(\Omega)$.*

*Proof.* In fact, let $u \in D(\Delta_0^\Omega)$, $\Delta_0 u = v$. Extend $v$ by 0 and let $w = E_n * v$. Then $w \in C^1(\mathbb{R}^n)$ and $\Delta w = v$. Thus $h := u - w$ is harmonic on $\Omega$ and so $h \in C^\infty(\Omega)$. It follows that $u = h + w \in C^1(\Omega)$. ◻

**Lemma 5.2.7.** *$D(\Delta_0^\Omega) \subset H_0^1(\Omega)$.*

*Proof.* Let $u \in D(\Delta_0^\Omega)$, $\Delta_0 u = f$. For $k > 0$, $u_k := (u - 1/k)^+ \in C_c(\Omega)$. Since $u \in C^1(\Omega)$ by the previous lemma, it follows from Proposition 3.2.8 that $u_k \in H_0^1(\Omega)$. By hypothesis, we have

$$\int_\Omega \nabla u \nabla v = -\int_\Omega f v \quad (v \in \mathcal{D}(\Omega)).$$

Taking $v = u_k$, we obtain

$$\int_\Omega |\nabla u_k|^2 = \int_\Omega \nabla u \nabla u_k = -\int_\Omega f u_k \leq \|f\|_{L^2(\Omega)} \|u_k\|_{L^2(\Omega)}.$$

It follows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Let $(u_{k_m})_{m\in\mathbb{N}}$ be a subsequence which converges weakly in $H_0^1(\Omega)$. Since $u_{k_m} \to u$ in $L^2(\Omega)$, it follows that this weak limit is $u$. Hence $u \in H_0^1(\Omega)$. ◻

**Proposition 5.2.8.** *Suppose that $\Omega$ is Dirichlet regular. Then*

$$e^{t\Delta_0^\Omega} = e^{t\Delta_\Omega^D}{}_{|C_0(\Omega)} \quad (t \geq 0). \tag{5.7}$$

*In particular, $\left(e^{t\Delta_0^\Omega}\right)_{t\geq 0}$ is positive and contractive.*

*Proof.* Because of Euler's Formula, it suffices to show that $R(\lambda, \Delta_0^\Omega)f = R(\lambda, \Delta_\Omega^D)f$ for all $f \in C_0(\Omega)$. Let $u = R(\lambda, \Delta_0^\Omega)f$, where $f \in L^2(\Omega)$. Then $u \in D(\Delta_0^\Omega) \subset H_0^1(\Omega)$ and $\lambda u - \Delta u = f$ in $\mathcal{D}(\Omega)'$. It follows that

$$\int_\Omega \lambda uv + \int_\Omega \nabla u \nabla v = \int_\Omega fv$$

for all $v \in \mathcal{D}(\Omega)$ and hence for all $v \in H_0^1(\Omega)$ by density. This implies that $u \in D(\Delta_\Omega^D)$ and $\lambda u - \Delta_\Omega^D u = f$. This proves the claim.

Now positivity follows from Theorem 3.3.1. It follows from Theorem 4.2.4 that the semigroup $\left(e^{t\Delta_0^\Omega}\right)_{t\geq 0}$ is contractive in $C_0(\Omega)$. □

## 5.3 Continuity of the kernel at the boundary

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set which is Dirichlet regular. We now want to consider the heat equation on $\Omega$ with non-autonomous boundary conditions. Let $\tau > 0$. Consider the **parabolic cylinder**

$$\Omega_\tau := (0, \tau) \times \Omega$$

and denote by

$$\Gamma_\tau := (\{0\} \times \overline{\Omega}) \cup ([0, \tau) \times \Gamma)$$

the **parabolic boundary** of $\Omega_\tau$.

We may visualise $\Omega_\tau$ as follows. Let $n = 2$ and draw $\Omega \subset \mathbb{R}^2$ in the plane and trace the time $t$ on the vertical axis. Then $\Omega_\tau$ is a cylinder of height $\tau$. Its parabolic boundary $\Gamma_\tau$ is the topological boundary without the top $\Gamma_\tau = \partial\Omega_\tau \setminus (\{\tau\} \times \overline{\Omega})$.

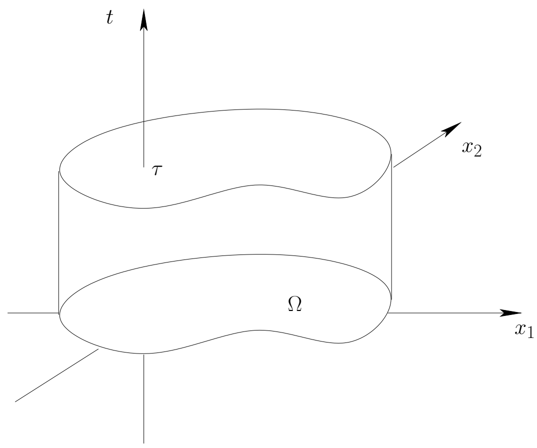


Figure 5.2: Parabolic cylinder.

**Theorem 5.3.1.** *Let $\psi \in C(\Gamma_\tau)$. Then there exists a unique solution of the* **Heat-Boundary Value Problem**

$$(H_\psi) \qquad \begin{cases} u \in C(\overline{\Omega}_\tau) \cap C^\infty(\Omega_\tau), \\ u_{|\Gamma_\tau} = \psi, \\ u_t - \Delta u = 0 \quad \text{in } \Omega_\tau. \end{cases}$$

We refer to [ABHN01, Theorem 6.2.8 p.407] for the proof of existence. Uniqueness follows from the Parabolic Maximum Principle.

**Proposition 5.3.2** (Parabolic Maximum Principle)**.** *Let $u \in C(\overline{\Omega}_\tau) \cap C^\infty(\Omega_\tau)$ such that*

$$u_t - \Delta u = 0 \quad \text{in } \Omega_\tau.$$

*Then*

$$\min_{\Gamma_\tau} u \le u(t,x) \le \max_{\Gamma_\tau} u$$

*for all $0 \le t \le \tau$, $x \in \overline{\Omega}$.*

*Proof.* Assume that there exists $0 < t_0 \le \tau$, $x_0 \in \Omega$ such that $u(t_0,x_0) > \max_{\Gamma_\tau} u$. Let $v(t,x) = u(t,x) - \varepsilon t$, where $\varepsilon > 0$ is so small that

$$u(t_0,x_0) - \varepsilon t_0 > \max_{\Gamma_\tau} u + \tau\varepsilon.$$

Then $v(t_0,x_0) > \max_{\Gamma_\tau} v$. Hence there exist $t_1 \in (0,\tau]$, $x_1 \in \Omega$ such that $v(t_1,x_1) = \max_{\overline{\Omega}_\tau} v$. Consequently, $v_t(t_1,x_1) \ge 0$, $\Delta v(t_1,x_1) \le 0$. Hence

$$0 \le v_t(t_1,x_1) = u_t(t_1,x_1) - \varepsilon = \Delta u(t_1,x_1) - \varepsilon = \Delta v(t_1,x_1) - \varepsilon \le -\varepsilon,$$

which is a contradiction. □

Consider the Gaussian kernel given by $k^{\mathbb{R}^n}(t,x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$. Then $k^{\mathbb{R}^n} \in C^\infty((0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n)$ and

$$k_t^{\mathbb{R}^n} - \Delta_x k^{\mathbb{R}^n} = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n.$$

The Gaussian Semigroup $G_0$ on $C_0(\mathbb{R}^n)$ is given by

$$(G_0(t)f)(x) = \int_{\mathbb{R}^n} k^{\mathbb{R}^n}(t,x,y) f(y) dy.$$

We have $\lim_{t\downarrow 0} G_0(t)f = f$ in $C_0(\mathbb{R}^n)$. Note that the kernel $k^{\mathbb{R}^n}$ has a singularity at $(0,y,y)$ for all $y \in \mathbb{R}^n$. But if $y \in \Omega$, then

$$\lim_{t\downarrow 0} k^{\mathbb{R}^n}(t,x,y) = 0 \tag{5.8}$$

uniformly in $x \in \Gamma$.

In Section 4.2, we obtained the kernel $k^\Omega$ of the semigroup $e^{t\Delta_\Omega^D}$ by an abstract argument. Now we will construct the kernel by solving the Heat-Boundary Value Problem for special boundary values.

Let $\tau > 0$. For $y \in \Omega$ consider the function $\psi^y \in C(\Gamma_\tau)$ given by

$$\psi^y(t,x) = \begin{cases} k^{\mathbb{R}^n}(t,x,y) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

By Theorem 5.3.1, there exists a unique function

$$p(\cdot,\cdot,y) \in C(\overline{\Omega}_\tau) \times C^\infty(\Omega_\tau)$$

such that $p(\cdot,\cdot,y) = \psi^y$ on $\Gamma_\tau$ and $u_t - \Delta u = 0$ in $\Omega_\tau$.

**Lemma 5.3.3.** *One has $p \in C([0,\tau] \times \overline{\Omega} \times \Omega)$.*

*Proof.* Let $y_k \to y$ in $\Omega$ as $k \to \infty$. Then $\psi^{y_k} \to \psi^y$ in $C(\Gamma_\tau)$. It follows from the Parabolic Maximum Principle, Proposition 5.3.2, that $p(\cdot,\cdot,y_k) \to p(\cdot,\cdot,y)$ in $C(\overline{\Omega}_\tau)$ as $k \to \infty$. $\square$

Since $\tau > 0$ is arbitrary, and because of the uniqueness of the solutions of $(H_\psi)$, we find $p \in C(\mathbb{R}_+ \times \overline{\Omega} \times \Omega)$ satisfying

$$p(t,x,y) = k^{\mathbb{R}^n}(t,x,y) \qquad \text{for } x \in \Gamma,\ t > 0,\ y \in \Omega; \tag{5.9}$$

$$p(0,x,y) = 0 \qquad \text{for } x \in \overline{\Omega},\ y \in \Omega; \tag{5.10}$$

$$p(\cdot,x,y) \in C^\infty(0,\infty) \qquad \text{for } x, y \in \Omega; \tag{5.11}$$

$$p(t,\cdot,y) \in C^\infty(\Omega) \qquad \text{for } t > 0,\ y \in \Omega; \tag{5.12}$$

$$p_t(t,x,y) = \Delta_x p(t,x,y) \qquad (x, y \in \Omega,\ t > 0). \tag{5.13}$$

Let $k(t,x,y) = k^{\mathbb{R}^n}(t,x,y) - p(t,x,y)$. Then $k \in C(\mathbb{R}_+ \times \overline{\Omega} \times \Omega)$ satisfies

$$k(t,x,y) = 0 \qquad \text{for } t \geq 0,\ x \in \Gamma,\ y \in \Omega; \tag{5.14}$$

$$k(\cdot,x,y) \in C^\infty(0,\infty) \qquad \text{for } x, y \in \Omega; \tag{5.15}$$

$$k(t,\cdot,y) \in C^\infty(\Omega) \qquad \text{for } t > 0,\ y \in \Omega; \tag{5.16}$$

$$k_t(t,x,y) = \Delta_x k(t,x,y) \qquad t > 0,\ x, y \in \Omega. \tag{5.17}$$

For $u_0 \in \mathcal{D}(\Omega)$, let

$$u(t,x) = \int_\Omega k(t,x,y) u_0(y) dy.$$

Then $u \in C(\mathbb{R}_+ \times \overline{\Omega})$ satisfies

$$u(t,\cdot) \in C_0(\Omega) \qquad t > 0; \tag{5.18}$$

$$\lim_{t \downarrow 0} u(t,\cdot) = u_0 \qquad \text{in } C_0(\Omega); \tag{5.19}$$

$$u \in C^\infty((0,\infty) \times \Omega); \tag{5.20}$$

$$u_t = \Delta u \qquad \text{in } (0,\infty) \times \Omega. \tag{5.21}$$

It follows that

$$u(t,\cdot) = e^{t\Delta_0^\Omega} u_0. \tag{5.22}$$

We show the details of this below, but first we establish our main result of this section.

Since $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$, we deduce from (5.22) that $k$ is the kernel of $e^{t\Delta_\Omega^D}$. We denote this kernel by $k^\Omega$ in the following. Since $e^{t\Delta_\Omega^D}$ is self-adjoint, we finally obtain the following result.

**Theorem 5.3.4** (Regularity of the kernel of $e^{t\Delta_\Omega^D}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and Dirichlet regular. Then the kernel $k^\Omega$ of $e^{t\Delta_\Omega^D}$ can be chosen such that*

$$\begin{aligned} &k^\Omega \in C((0,\infty) \times \overline{\Omega} \times \overline{\Omega}), \\ &k^\Omega(t,x,y) = k^\Omega(t,y,x) && (x, y \in \overline{\Omega},\ t > 0), \\ &k^\Omega(t,x,y) = 0 && \text{if } x \in \Gamma, \\ &k^\Omega \in C^\infty((0,\infty) \times \Omega \times \Omega). \end{aligned}$$

*Moreover*

$$0 \le k^\Omega(t,x,y) \le (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

*for all $t > 0$, $x, y \in \overline{\Omega}$.*

**Corollary 5.3.5.** *If $\Omega$ is a Dirichlet regular bounded open set in $\mathbb{R}^n$, then $e^{t\Delta_0^\Omega}$ is a compact operator on $C_0(\Omega)$ for $t > 0$.*

This follows from the Arzelà-Ascoli Theorem.

*Proof of* (5.22). It can be shown that the function $v(t,x) = ((e^{t\Delta_0^\Omega})u_0)(x)$ is regular, namely, $v \in C^\infty((0,\infty) \times \Omega)$ (see [ISEM99/00]). Then (5.22) follows from the Parabolic Maximum Principle, Proposition 5.3.2. So far, we only know that

$$v(\cdot,x) \in C^\infty(0,\infty) \quad \text{for all } x \in \overline{\Omega}, \tag{5.23}$$

because $\left(e^{t\Delta_\Omega^D}\right)_{t\ge 0}$ is a holomorphic semigroup, and

$$v_t(t,x) = \Delta v(t,x) \quad (t > 0, x \in \Omega), \tag{5.24}$$

where the Laplacian is merely defined in the sense of distributions. On the basis of this more restircted information, we can argue as follows. Let $w(t,x) = u(t,x) - v(t,x)$. We want to show that $w \equiv 0$. Let $\tau > 0$. Then $w \in C(\overline{\Omega}_\tau)$, $w_{|\Gamma_\tau} = 0$. Assume that $\max_{\Omega_\tau} w > 0$ (otherwise consider $-w$ instead of $w$). Let $\varepsilon > 0$ be so small that also $w_1(t,x) = w(t,x) - \varepsilon t$ has stirctly positive maximum at some point $(t_1, x_1) \in (0,\tau] \times \Omega$. It follows from the following lemma applied to $f(x) = w_1(t_1, x)$ that there exists $x_0 \in \Omega$ such that $w_1(t_1, x_0) = w_1(t_1, x_1) = \max_{\Omega_\tau} w_1$ and $\Delta w_1(t_1, x_0) \le 0$. Since $w_1(\cdot, x_0) \in C^\infty(0,\infty)$ has a maximum in $t_1$, $\frac{d}{dt} w_1(t_1, x_0) \ge 0$. Then

$$0 \ge \Delta w_1(t_1, x_0) = \Delta w(t_1, x_0) = w_t(t_1, x_0) = \frac{d}{dt} w_1(t_1, x_0) + \varepsilon \ge \varepsilon,$$

a contradiction. □

**Lemma 5.3.6.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $f \in C_0(\Omega)$ such that $\Delta f \in C(\Omega)$ (where $\Delta f$ is understood in the sense of ditributions). Assume that $\max_\Omega f > 0$. Then there exists $x_0 \in \Omega$ such that $f(x_0) = \max_\Omega f$ and $\Delta f(x_0) \le 0$.*

*Proof.* The proof is similar to the one given for Proposition 5.2.4. Let $0 < \varepsilon < \max_\Omega f$, $U = \{x \in \overline{\Omega} : f(x) < \varepsilon/3\}$, $K = \{x \in \Omega : f(x) \ge 2\varepsilon/3\}$, $\delta = \operatorname{dist}(K, \overline{U})$. Let $f_k = \rho_k * f$, $k_0 > 1/\delta$. Then for $k \ge k_0$, $f_k(x) \le 2\varepsilon/3$ for all $x \in U$. The set $K_1 = \Omega \setminus U$ is compact. Hence $f_k \to f$ uniformly in $K_1$. Consequently, for sufficiently large $k$, $f_k$ has a maximum at a point $x_k \in K_1$. We may assume that $x_k \to x_0 \in K_1$ as $k \to \infty$ (taking a subsequence otherwise). Then

$$f(x_0) = \lim_{k\to\infty} f_k(x_k) = \lim_{k\to\infty} \|f_k\|_{C(K_1)} = \|f\|_{C(K_1)} = \max_\Omega f.$$

Moreover, since $f_k \in C^\infty(\Omega)$ and $f_k(x_k)$ is a maximum, it follows that $\Delta f_k(x_k) \le 0$. Hence

$$f(x_0) = \lim_{k\to\infty} (\rho_k * \Delta f)(x_k) = \lim_{k\to\infty} \Delta f_k(x_k) \le 0.$$

□

## 5.4 Exercises

In the following $\Omega$ is a bounded, open subset of $\mathbb{R}^n$ whose boundary we denote by $\Gamma$. Firstly, we give an alternative proof of Corollary 5.3.5.

**Exercise 5.4.1** (Compactness of $e^{t\Delta_\Omega^D}$). *Assume that $\Omega$ is Dirichlet regular.*

*a) Show that $R(0, \Delta_\Omega^D) L^\infty(\Omega) \subset C_0(\Omega)$.*

**Hint:** Let $f \in L^\infty(\Omega)$. Then $v = E_n * f \in C(\mathbb{R}^n)$. Proceed as in part a) of Theorem 5.2.3 to find $u \in C_0(\Omega)$ such that $\Delta u = f$.

b) *Show that* $e^{t\Delta_\Omega^D} L^2(\Omega) \subset C_0(\Omega)$ *for* $t > 0$.

**Hint:** $e^{t\Delta_\Omega^D} L^2(\Omega) \subset L^\infty(\Omega)$ by Theorem 4.2.1,

$$e^{t\Delta_\Omega^D} = -R(0, \Delta_\Omega^D) e^{\frac{t}{2}\Delta_\Omega^D} \Delta_\Omega^D e^{\frac{t}{2}\Delta_\Omega^D}.$$

c) *Deduce from b) that* $e^{t\Delta_\Omega^D}$ *is compact for* $t > 0$.

**Exercise 5.4.2** (Consistency of the spectra)**.** *Assume that* $\Omega$ *is Dirichlet regular.*

a) *Show that* $\Delta_0^\Omega$ *has compact resolvent. Use Exercise 5.4.1, or Corollary 5.3.5.*

b) *Show that* $\sigma(\Delta_0^\Omega) = \sigma(\Delta_\Omega^D)$.

**Exercise 5.4.3** (Necessity of Dirichlet regularity)**.** *Assume that* $\Delta_0^\Omega$ *is surjective.*

a) *Show that* $0 \in \rho(\Delta_0^\Omega)$.

b) *Show that for each* $\varphi \in F = \{\phi_{|\Gamma} : \phi \in C^2(\mathbb{R}^2)\}$, *there exists a solution of* $(D_\varphi)$.

c) *Show that* $\Omega$ *is Dirichlet regular.*

**Hint:** By Stone-Weierstraß Theorem, the space $F$ is dense in $C(\Gamma)$. Use (5.2) and b).

**Exercise 5.4.4** (Elliptic Maximum Principle)**.** *Let* $h \in C(\overline{\Omega})$ *be harmonic in* $\Omega$. *Show that* $\max_{\overline{\Omega}} h = \max h_{|\Gamma}$.

**Hint:** Assuming that $\max_{\overline{\Omega}} h > \max h_{|\Gamma}$, show that $u(x) = h(x) + \varepsilon|x|^2$ has a maximum at some point $x_0 \in \Omega$. Hence $\Delta u(x_0) \leq 0$.

## 5.5 Comments

Firstly, we give some additional information concerning Section 5.2. The Maximum Principle, Proposition 5.2.4, is due to Lumer–Paquet [LP76]. In [ABHN01, Chapter 6], a different approach is chosen based on resolvent-positive operators. For a proof of Examples 5.2.2 see [DL88, Chapter II] concerning a), c) and d), and see [Con78] for b).

Here is some further information.

### 5.5.1 Characterisation of Dirichlet regularity

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. The following assertions are equivalent:

(*i*) $\Omega$ is Dirichlet regular;

(*ii*) $\rho(\Delta_0^\Omega) \neq \emptyset$;

(*iii*) the eigenfunction corresponding to the first eigenvalue of $-\Delta_\Omega^D$ is in $C_0(\Omega)$;

(*iv*) $D(\Delta_p^\Omega) \subset C_0(\Omega)$ for all $p > n/2$.

We refer to [ArBe99] for the proofs.

### 5.5.2 Continuity of the kernel at the boundary

Theorem 5.3.4 is known for regular open sets. We were inspired by Dodziuk's presentation of the eigenvalue distribution [Dod81], where it is assumed that $\Omega$ is of class $C^2$. The proof of Theorem 5.3.4 is based on Theorem 5.3.1 which goes back to [Are00]. So we finally have the following characterisation of the continuity at the boundary, which might be new.

**Theorem 5.5.1.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary $\Gamma$, and let $k^\Omega$ be the kernel of $e^{t\Delta_\Omega^D}$. The following are equivalent:*

(*i*) *$k^\Omega \in C((0,\infty) \times \overline{\Omega} \times \overline{\Omega})$ and $k^\Omega(t,x,y) = 0$ if $x \in \Gamma$ or $y \in \Gamma$;*

(*ii*) *$\Omega$ is Dirichlet regular.*

*Proof.* The implication $(ii) \Rightarrow (i)$ has been proved in the text. Assume $(i)$. Denote by $u_1$ the first eigenvector of $-\Delta_\Omega^D$ with corresponding eigenvalue $\lambda_1$. Then

$$\big(e^{-t\lambda_1}u_1\big)(x) = \big(e^{t\Delta_\Omega^D}u_1\big)(x) = \int_\Omega k^\Omega(t,x,y)u_1(y)dy.$$

Hence $u_1 \in C_0(\Omega)$. It follows from 5.5.1 above that $\Omega$ is Dirichlet regular. □

### 5.5.3 Strict positivity of the kernel

Let $\Omega$ be open, bounded and Dirichlet regular. It follows from the Strong Maximum Principle [Eva98, 2.3.3] and (5.20) and (5.21) that $k^\Omega(t,x,y) > 0$ for all $x, y \in \Omega$.

*Dedicated to* Rainer Nagel *on*
*the occasion of his 65$^{th}$ birthday*

# Lecture 6

# Weyl's Theorem

The aim of this lecture is to prove Weyl's famous result on the asymptotic distribution of the eigenvalues of the Dirichlet Laplacian. This can be achieved with the help of the heat kernel. First, we will show how the kernel can be decomposed by the eigenfunctions (Mercer's Theorem). For this, we need the continuity of the kernel up to the boundary, which we have established in Lecture 5. Then we may express the trace of the heat semigroup with the help of the kernel. The upper limit is a simple comparison with the Gaussian kernel. Here we use the monotonicity of the kernel as a function of the domain (Lecture 4). The lower estimate is more subtle. We will need to prove that for small time the kernel is not affected by the boundary. Once the trace is estimated, Weyl's result follows by an application of Karamata's Tauberian Theorem. This lecture is divided into three sections.



## 6.1 Mercer's Theorem

In this section, we show how a continuous symmetric kernel can be decomposed into eigenfunctions. In particular, we obtain the trace of the operator in terms of the kernel, namely as the integral over the diagonal of the kernel. Throughout this section, the underlying field may be $\mathbb{R}$ or $\mathbb{C}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $k \in C(\overline{\Omega} \times \overline{\Omega})$ be a symmetric kernel, i.e.,

$$k(x, y) = \overline{k(y, x)} \quad (x, y \in \overline{\Omega}).$$

Define the operator $B_k$ on $L^2(\Omega)$ by

$$(B_kf)(x) = \int_\Omega k(x,y)f(y)dy. \tag{6.1}$$

Then $B_k$ is a selfadjoint, compact operator on $L^2(\Omega)$. In fact, $B_k$ is a Hilbert–Schmidt operator. One may also use the Arzelà–Ascoli Theorem to show that $B_k$ maps the unit ball of $L^2(\Omega)$ into a compact set in $C(\overline{\Omega})$.

By the Spectral Theorem (Theorem 1.4.8), there exists an orthonormal basis $\{e_j : j \in J\}$ of $L^2(\Omega)$ and $\lambda_j \in \mathbb{R} \setminus \{0\}$ such that

$$B_kf = \sum_{j\in J} \lambda_j \,(f \mid e_j)\, e_j \qquad (f \in L^2(\Omega)). \tag{6.2}$$

Here $J$ is finite or $J = \mathbb{N}$, and the series (6.2) converges in $L^2(\Omega)$. Moreover, $\lim_{j\to\infty} \lambda_j = 0$ if $J = \mathbb{N}$. Since by the Dominated Convergence Theorem,

$$B_kL^2(\Omega) \subset C(\overline{\Omega}), \tag{6.3}$$

it follows that $e_m \in C(\overline{\Omega})$ for all $m \in J$.

A bounded operator $B$ on a Hilbert space $H$ is called **form-positive** if $B = B^*$ and $(Bf \mid f) \geq 0$ for all $f \in H$. If $B$ is symmetric and compact, then by the Spectral Theorem (Theorem 1.4.8) this is equivalent to the fact that all its eigenvalues are non-negative. We want to prove the following theorem, where we assume that $J = \mathbb{N}$ (the finite case being much simpler).

**Theorem 6.1.1** (Mercer)**.** *Assume that $B_k$ is form-positive. Then*

$$k(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)} \quad (x, y \in \overline{\Omega}), \tag{6.4}$$

*where the series converges absolutely and uniformly in $\overline{\Omega} \times \overline{\Omega}$.*

We need some preparation. In the first lemma, the assumption that $B_k$ is form-positive is not needed. Recall that $B_kL^2(\Omega) \subset C(\overline{\Omega})$.

**Lemma 6.1.2.** *Let $f \in L^2(\Omega)$. Then*

$$(B_kf)(x) = \sum_{j=1}^{\infty} \lambda_j \,(f \mid e_j)\, e_j(x), \tag{6.5}$$

*where the series converges absolutely and uniformly in $x \in \overline{\Omega}$.*

*Proof.* Let $x \in \overline{\Omega}$. Then by (6.2),

$$\sum_{j=1}^{\infty} |\lambda_j e_j(x)|^2 = \sum_{j=1}^{\infty} |(B_k e_j)(x)|^2 = \sum_{j=1}^{\infty} \left| \int_\Omega k(x,y) e_j(y) dy \right|^2$$

$$= \sum_{j=1}^{\infty} \left| \, (k(x,\cdot) \,|\, e_j) \, \right|^2 \leq \|k(x,\cdot)\|^2_{L^2(\Omega)} \leq |\Omega| \|k\|^2_{L^\infty(\Omega\times\Omega)},$$

where we have used **Bessel's Inequality**.

Let $f \in L^2(\Omega)$. Then for $N \in \mathbb{N}$, $x \in \overline{\Omega}$, we get by the Cauchy–Schwarz Inequality

$$\sum_{j \geq N} \left| \lambda_j \, (f \,|\, e_j) \, e_j(x) \right| \leq \left( \sum_{j\geq N} |\lambda_j e_j(x)|^2 \right)^{1/2} \cdot \left( \sum_{j \geq N} \left| \, (f \,|\, e_j) \, \right|^2 \right)^{1/2}$$

$$\leq |\Omega|^{1/2} \|k\|_{L^\infty(\Omega\times\Omega)} \left( \sum_{j\geq N} \left| \, (f \,|\, e_j) \, \right|^2 \right)^{1/2} \longrightarrow 0$$

as $N \to \infty$ uniformly in $x \in \overline{\Omega}$. We have shown that the series converges absolutely and uniformly in $x \in \overline{\Omega}$.

Since the function $B_k f$ is continuous and since the identity (6.5) is valid in $L^2(\Omega)$, it also holds for all $x \in \overline{\Omega}$. □

Next we show that under the assumptions of Mercer's Theorem, the kernel has a positive diagonal.

**Lemma 6.1.3.** *Assume that $B_k$ is form-positive. Then $k(x,x) \geq 0$ for all $x \in \overline{\Omega}$.*

*Proof.* Let $x_0 \in \overline{\Omega}$ and let $f_k(y) = \rho_k(x_0 - y)$, where $\rho_k$ are the mollifiers from the proof of Proposition 3.2.8. Then

$$0 \leq (B_k f_m \,|\, f_m) = \int_\Omega \int_\Omega k(x,y) f_m(x) \overline{f_m(y)} \, dy \, dx \longrightarrow k(x_0,x_0) \quad \text{as } m \to \infty,$$

since $\operatorname{supp} f_m \subset B(x_0, 1/m)$ and

$$\int_{\mathbb{R}^n} f_m(x) dx = 1.$$

□

*Proof of Theorem 6.1.1.* For $m \in \mathbb{N}$, let

$$k_m(x,y) = \sum_{j=1}^{m} \lambda_j e_j(x) \overline{e_j(y)}.$$

Then $k_m \in C(\overline{\Omega} \times \overline{\Omega})$. Moreover,

$$(B_{k-k_m} f \mid f) = (B_k f \mid f) - (B_{k_m} f \mid f) = \sum_{j>m} \lambda_j \, (f \mid e_j) \, (e_j \mid f) = \sum_{j>m} \lambda_j | \, (f \mid e_j) \, |^2 \geq 0.$$

It follows from Lemma 6.1.3 that $k(x,x) - k_m(x,x) \geq 0$ for all $x \in \overline{\Omega}$. Consequently,

$$\sum_{j=1}^{m} \lambda_j |e_j(x)|^2 = k_m(x,x) \leq k(x,x)$$

for all $m \in \mathbb{N}$, $x \in \overline{\Omega}$. Hence

$$\sum_{j=1}^{m} \lambda_j |e_j(x)|^2 \leq \|k\|_{L^\infty(\Omega\times\Omega)} \tag{6.6}$$

for all $x \in \overline{\Omega}$.

By the Cauchy–Schwarz Inequality,

$$\sum_{j=1}^{m} |\lambda e_j(x)\overline{e_j(y)}|^2 \leq \left(\sum_{j=1}^{m} \lambda_j |e_j(x)|^2\right)^{1/2} \cdot \left(\sum_{j=1}^{m} \lambda_j |e_j(y)|^2\right)^{1/2} \leq \|k\|_{L^\infty(\Omega\times\Omega)}. \tag{6.7}$$

Thus the series

$$\tilde{k}(x,y) := \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)} \tag{6.8}$$

converges absolutely for $x, y \in \overline{\Omega}$. We want to show that the convergence is uniform on $\overline{\Omega} \times \overline{\Omega}$.

First we fix $x \in \overline{\Omega}$ and show uniform convergence in $y \in \overline{\Omega}$. Let $\varepsilon > 0$. Then by (6.6), there exists $N(x) \in \mathbb{N}$ such that

$$\sum_{j \geq N(x)} \lambda_j |e_j(x)|^2 \leq \varepsilon. \tag{6.9}$$

It follows (6.6) and (6.9) that

$$\sum_{j \geq N(x)} \left|\lambda_j e_j(x)\overline{e_j(y)}\right| \leq \left(\sum_{j \geq N(x)} \lambda_j |e_j(x)|^2\right)^{1/2} \cdot \left(\sum_{j \geq N(x)} \lambda_j |e_j(y)|^2\right)^{1/2} \leq \varepsilon \|k\|^{1/2}_{L^\infty(\Omega\times\Omega)} \tag{6.10}$$

for all $y \in \overline{\Omega}$. Therefore $\tilde{k}(x,\cdot)$ is continuous.

We now show that $k = \tilde{k}$. Let $x \in \overline{\Omega}$. Then

$$h(y) = k(x,y) - \tilde{k}(x,y)$$

defines a continuous function on $\overline{\Omega}$. For each $f \in C(\overline{\Omega})$, we have

$$\int_\Omega h(y)f(y)dy = \int_\Omega k(x,y)f(y)dy - \int_\Omega \tilde{k}(x,y)f(y)dy$$
$$= (B_k f)(x) - \sum_{j=1}^{\infty} \lambda_j e_j(x)\,(f \mid e_j) = 0$$

by Lemma 6.1.2. This implies that $h \equiv 0$.

It follows from (6.8) that

$$k(x,x) = \sum_{j=1}^{\infty} \lambda_j |e_j(x)|^2 \quad (x \in \overline{\Omega}). \tag{6.11}$$

This series converges uniformly by Dini's Theorem. This shows in particular that the number $N(x)$ in (6.9) can be chosen independently of $x \in \overline{\Omega}$. Now as a consequence of (6.10), the series (6.4) converges absolutely and uniformly to $k(x,y)$. ☐

**Corollary 6.1.4.** *Assume that $B_k$ is form-positive. Then the trace of $B_k$ is given by*

$$\operatorname{Tr}(B_k) := \sum_{j=1}^{\infty} \lambda_j = \int_\Omega k(x,x)dx. \tag{6.12}$$

## 6.2 A Tauberian theorem

Let $\mu$ be a positive Borel measure on $[0,\infty)$ such that

$$\hat{\mu}(t) := \int_0^\infty e^{-tx} d\mu(x) < \infty \tag{6.13}$$

for all $t > 0$. The function $\hat{\mu} : (0,\infty) \to \mathbb{R}$ is called the **Laplace Transform** of $\mu$. Our aim is to prove the following theorem relating the asymptotic behaviour of $\mu([0,x])$ as $x \to \infty$ to the asymptotic behaviour of $\hat{\mu}(t)$ as $t \downarrow 0$.

**Theorem 6.2.1** (Karamata)**.** *Let $r \geq 0$, $a \in \mathbb{R}$. The following are equivalent:*

*(i)* $\lim_{t \downarrow 0} t^r \hat{\mu}(t) = a$;

*(ii)* $\lim_{x \to \infty} x^{-r} \mu([0,x]) = \dfrac{a}{\Gamma(r+1)}$.

Here

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy \qquad (\alpha > 0)$$

is Euler's Gamma Function.

The implication $(ii) \Rightarrow (i)$, i.e. deducing properties of the transform $\hat{\mu}$ from the assumptions on $\mu$, is an **Abelian theorem**, which is easy to prove. The difficult part is the converse implication $(i) \Rightarrow (ii)$, which is a **Tauberian theorem**.

For the proof, we will use the fact that a bounded Borel measure $\nu$ on $[0, \infty)$ can be identified with a positive linear form $\Phi_\nu$ on

$$C_0[0, \infty) := \{f \in C[0, \infty) : \lim_{x \to \infty} f(x) = 0\}$$

by the relation

$$\Phi_\nu(f) = \int_0^\infty f(x) d\nu(x).$$

Note that $C_0[0, \infty)$ is a Banach space endowed with the supremum norm $\|\cdot\|_\infty$. Each positive linear form is continuous, and in fact

$$\|\Phi_\nu\| = \int_0^\infty 1 d\nu(x). \tag{6.14}$$

The assumption that the integral (6.13) converges means that the measure $e^{-tx} d\mu(x)$ is finite for all $t > 0$. By $C_c[0, \infty)$, we denote the space of those functions $f \in C_0[0, \infty)$ which vanish outside a set $[0, \tau]$ for some $\tau > 0$.

*Proof of Theorem 6.2.1.*

***(i)*⇒*(ii)* a)** The case $r = 0$ is simply an application of the Monotone Convergence Theorem. In fact,

$$\begin{aligned} a = \sup_{t>0} \int_{[0,\infty)} e^{-tx} d\mu(x) &= \sup_{t>0} \sup_{b>0} \int_{[0,b]} e^{-tx} d\mu(x) \\ &= \sup_{b>0} \sup_{t>0} \int_{[0,b]} e^{-tx} d\mu(x) = \sup_{b>0} \int_{[0,b]} d\mu(x) = \sup_{b>0} \mu([0, b]). \end{aligned}$$

**b)** Now we assume that $r > 0$. For $t > 0$, define the positive Borel measure $\mu_t$ by

$$\mu_t(A) = t^r \mu(t^{-1} A),$$

i.e., for $f \in C_c[0, \infty)$, we have

$$\int_{[0,\infty)} f(x) d\mu_t(x) = t^r \int_{[0,\infty)} f(xt) d\mu(x).$$

Let $\nu$ be the measure $x^{r-1}dx$. It follows from the assumption that for each $n \in \mathbb{N}_0$,

$$\lim_{t\downarrow 0} \int_{[0,\infty)} e^{-x} e^{-nx} d\mu_t(x) = \frac{a}{\Gamma(r)} \int_0^\infty e^{-x} e^{-nx} d\nu(x). \tag{6.15}$$

In fact,

$$\begin{aligned} \int_{[0,\infty)} e^{-x} e^{-nx} d\mu_t(x) &= t^r \int_{[0,\infty)} e^{-t(n+1)x} d\mu(x) = \frac{1}{(1+n)^r} \big(t(1+n)\big)^r \hat{\mu}\big(t(n+1)\big) \\ &\longrightarrow \frac{1}{(1+n)^r} a = \frac{a}{\Gamma(r)} \int_0^\infty e^{-x(1+n)} d\nu(x) \qquad \text{as } t \downarrow 0. \end{aligned}$$

Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\lim_{k\to\infty} t_k = 0$. It follows from (6.15) applied to $n = 0$ that

$$\sup_{k\in\mathbb{N}} \|e^{-x} d\mu_{t_k}\| < \infty.$$

The functions of the form $e^{-nx}$ $(n \in \mathbb{N})$ are total in $C_0[0,\infty)$ by the Stone–Weierstraß Theorem. Thus it follows from (6.15) that

$$\lim_{k\to\infty} \int_{[0,\infty)} f(x) e^{-x} d\mu_{t_k} = \frac{a}{\Gamma(r)} \int_0^\infty f(x) e^{-x} d\nu(x) \tag{6.16}$$

for all $f \in C_0[0,\infty)$. We will show in an instant that this implies that

$$\lim_{k\to\infty} \mu_{t_k}([0,1]) = \frac{a}{\Gamma(r)} \nu([0,1]). \tag{6.17}$$

Since

$$\mu_{t_k}([0,1]) = t_k^{-r} \int_{[0,\infty)} 1_{[0,1]}(t_k x) d\mu(x) = t_k^{-r} \mu([0,1/t_k]),$$

and

$$\nu([0,1]) = \int_0^1 x^{r-1} dx = \frac{1}{r} \quad \text{and} \quad r\Gamma(r) = \Gamma(1+r),$$

then (6.17) implies claim $(ii)$.

In order to prove (6.17), we first observe that (6.16) implies that

$$\lim_{k\to\infty} \int_{[0,\infty)} g(x) d\mu_{t_k}(x) = \left( \int_0^\infty g(x) d\nu(x) \right) \cdot \frac{a}{\Gamma(r)} \tag{6.18}$$

for all $g \in C_c[0,\infty)$.

Now let $\varepsilon > 0$. Then there exists $g_\varepsilon \in C_c[0,\infty)$ such that $g_\varepsilon \geq 0$, $g_\varepsilon = 1$ on $[0,1]$ and

$$\int_0^\infty g_\varepsilon(x)d\nu(x) \geq \nu([0,1]) \geq \int_0^\infty g_\varepsilon(x)d\nu(x) - \varepsilon.$$

Hence by (6.18),

$$\varlimsup_{k\to\infty} \mu_{t_k}([0,1)) \leq \varlimsup_{k\to\infty} \int_{[0,\infty)} g_\varepsilon(x)d\mu_{t_k} = \frac{a}{\Gamma(r)} \int_0^\infty g_\varepsilon(x)d\nu(x) \leq \frac{a}{\Gamma(r)}\big(\nu([0,1]) + \varepsilon\big).$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\varlimsup_{k\to\infty} \mu_{t_k}([0,1]) \leq \frac{a}{\Gamma(r)}\nu([0,1]).$$

Similarly, there exists $h_\varepsilon \in C_c[0,\infty)$ such that $0 \leq h_\varepsilon \leq 1$ on $[0,\infty)$, $\operatorname{supp} h_\varepsilon \subset [0,1]$ and

$$\int_0^\infty h_\varepsilon(x)d\nu(x) \geq \nu([0,1]) - \varepsilon.$$

Hence by (6.18),

$$\varliminf_{k\to\infty} \mu_{t_k}([0,1]) \geq \varliminf_{k\to\infty} \int_{[0,\infty)} h_\varepsilon(x)d\mu_{t_k}(x) = \frac{a}{\Gamma(r)} \int_0^\infty h_\varepsilon(x)d\nu(x) \geq \frac{a}{\Gamma(r)}\big(\nu([0,1]) - \varepsilon\big).$$

Again since $\varepsilon > 0$ is arbitrary, it follows that

$$\varliminf_{k\to\infty} \mu_{t_k}([0,1]) \geq \frac{a}{\Gamma(r)}\nu([0,1])$$

Thus (6.17) is proved.

***(ii)*** $\Rightarrow$ ***(i)*** Let $\alpha(t) = \mu([0,t])$. Then $\alpha$ is increasing, and integration by parts yields

$$\begin{aligned}
t^r\hat{\alpha}(t) &= t^{r+1}\int_0^\infty e^{-xt}\alpha(x)dx \\
&= t^{r+1}\int_0^\infty e^{-y}\alpha\left(\frac{y}{t}\right)\frac{dy}{t} = t^r\int_0^\infty e^{-y}\left(1+\frac{y}{t}\right)^r\left(1+\frac{y}{t}\right)^{-r}\alpha\left(\frac{y}{t}\right)dy \\
&= \int_0^\infty e^{-y}(t+y)^r\left(1+\frac{y}{t}\right)^{-r}\alpha\left(\frac{y}{t}\right)dy \longrightarrow \int_0^\infty e^{-y}y^r\frac{a}{\Gamma(r+1)}dy = a
\end{aligned}$$

as $t \downarrow 0$ by the Dominated Convergence Theorem. The dominating function for $0 < t \leq 1$ is obtained by observing that $(1+x)^{-r}\alpha(x) \to a/\Gamma(r)$ as $x \to \infty$, and hence $C := \sup_{x>0}(1+x)^{-r}\alpha(x) < \infty$. Hence

$$e^{-y}(t+y)^r\left(1+\frac{y}{t}\right)^{-r}\alpha\left(\frac{y}{t}\right) \leq Ce^{-y}(1+y)^r$$

for $0 < t \leq 1$.

$\square$

## 6.3 Weyl's Formula

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. The Dirichlet Laplacian $\Delta_\Omega^D$ on $L^2(\Omega)$ is symmetric and m-dissipative and has compact resolvent by Corollary 4.2.5. Denote by

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

the eigenvalues of $-\Delta_\Omega^D$. Note that $\Delta_\Omega^D$ is injective (by the Poincaré Inequality, Theorem 3.4.1). Hence $\lambda_1 > 0$. Our aim is to investigate the asymptotic behaviour of $\lambda_n$ as $n \to \infty$. It follows from the Spectral Theorem that the eigenvalues of $e^{t\Delta_\Omega^D}$ are $e^{-\lambda_j t}$, $j \in \mathbb{N}$. Since $e^{t\Delta_\Omega^D}$ is a Hilbert–Schmidt operator,

$$\sum_{j=1}^{\infty} e^{-2\lambda_j t} < \infty, \quad \text{hence} \quad \sum_{j=1}^{\infty} e^{-\lambda_j t} < \infty$$

for $t > 0$. We denote by

$$\operatorname{Tr}\left(e^{t\Delta_\Omega^D}\right) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \tag{6.19}$$

the trace of $e^{t\Delta_\Omega^D}$. By $k^\Omega(t,\cdot,\cdot)$, we denote the kernel of $e^{t\Delta_\Omega^D}$. If $\Omega$ is Dirichlet regular, then by Theorem 5.3.4, the kernel $k^\Omega(t,\cdot,\cdot)$ is continuous on $\overline{\Omega} \times \overline{\Omega}$ and non-negative by Theorem 4.2.4. Therefore $e^{t\Delta_\Omega^D}$ is form-positive, hence by Mercer's Theorem (Theorem 6.1.1),

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_\Omega k^\Omega(t,x,x)dx. \tag{6.20}$$

We will use this formula to estimate the eigenvalues. An upper estimate follows from domination by the Gaussian kernel, Theorem 4.2.4, which implies that $k^\Omega(t,x,x) \le (4\pi t)^{-n/2}$ for all $x \in \overline{\Omega}$ and hence

$$0 \le \int_\Omega k^\Omega(t,x,x)dx \le \frac{|\Omega|}{(4\pi t)^{n/2}} \quad (t > 0). \tag{6.21}$$

We now proceed to give a lower estimate for small $t > 0$. As a first step, we give an estimate which may be interpreted physically. We may see $k^\Omega(t,x,y)$ as the amount of heat at the point $x$ at time $t$ if an initial heat of value 1 is concentrated at $y$. The same interpretation is also valid for

$$k^{\mathbb{R}^n}(t,x,y) = (4\pi t)^{-n/2} e^{-|x-y|/4t},$$

where the heat flow takes place in the entire space. The following estimate shows that for $x \in \Omega$, the heat amount $k^\Omega(t,x,y)$ is close to $k^{\mathbb{R}^n}(t,x,y)$ if $y \in \Omega$ is away from the boundary. Recall that $\Gamma = \partial\Omega$ is the boundary of $\Omega$.

**Proposition 6.3.1** (Kac's Principle: The boundary does not affect the kernel)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is Dirichlet regular. Let $x \in \Omega$ be fixed. For $y \in \Omega$, let*

$$t_0(y) = \frac{\operatorname{dist}(y, \Gamma)^2}{2n}.$$

*Then*

$$0 \le k^{\mathbb{R}^n}(t, x, y) - k^{\Omega}(t, x, y) \le \begin{cases} (4\pi t)^{-n/2} e^{-\operatorname{dist}(y,\Gamma)^2/4t} & \text{if } t \le t_0(y); \\ (4\pi t_0(y))^{-n/2} e^{-n/2} & \text{if } t > t_0(y). \end{cases}$$

*Proof.* Recall from (5.22) and its consequences that for $x, y \in \Omega$,

$$k^{\mathbb{R}^n}(t, x, y) - k^{\Omega}(t, x, y) = p(t, x, y),$$

where $p(\cdot, \cdot, y)$ solves the Heat-Boundary Value Problem with the following values on the parabolic boundary:

$$\begin{aligned} p(0, x, y) &= 0 && (x \in \overline{\Omega}); \\ p(t, x, y) &= (4\pi t)^{-n/2} e^{-|x-y|^2/4t} && (t > 0, x \in \Gamma). \end{aligned}$$

It follows from the Parabolic Maximum Principle that for $x, y \in \Omega$

$$p(t, x, y) \le \sup_{\substack{z \in \Gamma \\ 0 < s \le t}} (4\pi s)^{-n/2} e^{-|z-y|^2/4s} \le \sup_{0 < s \le t} (4\pi s)^{-n/2} e^{-\operatorname{dist}(y,\Gamma)^2/4s}.$$

The function

$$f(s) = (4\pi s)^{-n/2} e^{-\operatorname{dist}(y,\Gamma)^2/4s}$$

attains its maximum at $t_0(y)$ and is decreasing for $s > t_0(y)$ and increasing for $s < t_0(y)$. This proves the estimate. □

Using this proposition, we obtain the following lower estimate.

**Proposition 6.3.2.** *Assume that $\Omega$ is Dirichlet regular. Then*

$$\varliminf_{t \downarrow 0} t^{n/2} \int_{\Omega} k^{\Omega}(t, x, x) dx \ge \frac{|\Omega|}{(4\pi)^{n/2}}. \tag{6.22}$$

*Proof.* For $t > 0$, let

$$\begin{aligned} \Omega_1(t) &= \{x \in \Omega : \operatorname{dist}(x, \Gamma) \ge t^{1/4}\} \\ \Omega_2(t) &= \{x \in \Omega : \operatorname{dist}(x, \Gamma) < t^{1/4}\}. \end{aligned}$$

Then $\lim_{t\downarrow 0} |\Omega_2(t)| = 0$ by the Dominated Convergence Theorem. We will give an upper estimate of

$$t^{n/2} \int_{\Omega} \left(k^{\mathbb{R}^n}(t, x, x) - k^{\Omega}(t, x, x)\right) dx \qquad \text{as } t \downarrow 0.$$

First we consider the integral over $\Omega_1(t)$, where we use Proposition 6.3.1.

Let $t < (2n)^{-2}$. Then for $x \in \Omega_1(t)$,

$$\begin{aligned} t_0(x) = \frac{\operatorname{dist}(x,\Gamma)^2}{2n} &\geq \frac{t^{1/2}}{2n} > t \quad \text{and} \\ -\frac{\operatorname{dist}(x,\Gamma)^2}{4t} &\leq -\frac{1}{4t^{1/2}}. \end{aligned}$$

Hence by Proposition 6.3.1, for $x \in \Omega_1(t)$,

$$k^{\mathbb{R}^n}(t,x,x) - k^{\Omega}(t,x,x) \leq (4\pi t)^{-n/2} e^{-1/4t^{1/2}},$$

hence

$$t^{n/2} \int_{\Omega_1(t)} \left( k^{\mathbb{R}^n}(t,x,x) - k^{\Omega}(t,x,x) \right) dx \leq \frac{|\Omega|}{(4\pi)^{n/2}} e^{-1/4t^{1/2}} \to 0 \quad \text{as } t \downarrow 0.$$

On $\Omega_2(t)$, we estimate by the Gaussian kernel, since $k^{\Omega}(t,x,x) \geq 0$:

$$t^{n/2} \int_{\Omega_2(t)} \left( k^{\mathbb{R}^n}(t,x,x) - k^{\Omega}(t,x,x) \right) dx \leq t^{n/2} (4\pi t)^{-n/2} |\Omega_2(t)| = \frac{|\Omega_2(t)|}{(4\pi)^{n/2}} \to 0 \quad \text{as } t \downarrow 0.$$

These two estimates show that

$$\overline{\lim_{t\downarrow 0}} \left( \frac{|\Omega|}{(4\pi)^{n/2}} - \int_{\Omega} k^{\Omega}(t,x,x) dx \right) = \overline{\lim_{t\downarrow 0}}\, t^{n/2} \int_{\Omega} \left( k^{\mathbb{R}^n}(t,x,x) - k^{\Omega}(t,x,x) \right) dx = 0.$$

This implies (6.22). ☐

Thus by (6.21) and (6.22), we have proved that

$$\lim_{t\downarrow 0} t^{n/2} \int_{\Omega} k(t,x,x) dx = \frac{|\Omega|}{(4\pi)^{n/2}} \tag{6.23}$$

if $\Omega$ is Dirichlet regular.

Hence by (6.20), we deduce that

$$\lim_{t\downarrow 0} t^{n/2} \sum_{j=1}^{\infty} e^{-\lambda_j t} = \frac{|\Omega|}{(4\pi)^{n/2}}. \tag{6.24}$$

This is already a remarkable result. It shows in particular that the eigenvalues of the Dirichlet Laplacian $\Delta_{\Omega}^{D}$ determine the volume of $\Omega$. Now we use the Tauberian theorem of Section 6.2 to actually obtain Weyl's Formula.

For $\lambda > 0$, let $N(\lambda)$ be the number of $\lambda_j$ such that $\lambda_j \leq \lambda$.

**Theorem 6.3.3** (Weyl)**.** *Assume that $\Omega$ is Dirichlet regular. Then*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{n/2}}=\frac{\omega_n}{(2\pi)^n}|\Omega|, \tag{6.25}$$

*where $\omega_n=\pi^{n/2}/\Gamma(1+\frac{n}{2})$ denotes the volume of the unit ball in $\mathbb{R}^n$.*

*Proof.* Define the discrete measure $\mu$ on $\mathbb{R}_+$ by $\mu(\{\lambda\}):=\#\{j:\lambda=\lambda_j\}$. Then $\mu([0,\lambda])=N(\lambda)$ and

$$\hat{\mu}(t)=\int_0^\infty e^{-\lambda t}d\mu(\lambda)=\sum_{j=1}^\infty e^{-\lambda_j t}.$$

Thus by (6.24)

$$\lim_{t\downarrow 0}t^{n/2}\hat{\mu}(t)=\frac{|\Omega|}{(4\pi)^{n/2}}.$$

It follows from Karamata's Theorem (Theorem 6.2.1) that

$$\lim_{\lambda\to\infty}\lambda^{-n/2}N(\lambda)=\lim_{\lambda\to\infty}\lambda^{-n/2}\mu([0,\lambda])=\frac{1}{\Gamma(1+\frac{n}{2})}\frac{1}{(4\pi)^{n/2}}|\Omega|=\frac{\omega_n}{(2\pi)^n}|\Omega|.$$

□

## 6.4 Exercises

We first show by an example that in Mercer's Theorem the assumption that $B_k$ is form-positive cannot be omitted.

**Exercise 6.4.1** (Failure of the trace formula)**.** *Let $\Omega=(0,1)$. Consider the kernel $k(x,y)=|x-y|$. Show that the operator $B_k$ on $L^2(0,1)$ is not form-positive*

**Hint:** Use Mercer's Theorem.

**Exercise 6.4.2** (An example for the trace formula)**.** *Consider the kernel $k(x,y)=\min\{x,y\}$ on $[0,1]\times[0,1]$. Show that the operator $B_k$ on $L^2(0,1)$ is selfadjoint, compact and that $\sum_{j=1}^\infty\lambda_j=1/2$, where $\lambda_1\le\lambda_2\le\cdots$ are the eigenvalues of $B_k$ (repeated according to multiplicity).*

**Hint:** Show that $B_k$ is form-positive.

**Exercise 6.4.3** (Weyl's Formula on the interval)**.** *Let $\Omega=(0,\ell)$, $\ell\in\mathbb{R}$. Determine the eigenvalues of $\Delta_\Omega^D$ on $L^2(0,\ell)$. Prove Weyl's Formula (6.25) directly.*

**Exercise 6.4.4** (The Neumann Laplacian on an interval)**.** *Consider $\Delta_\Omega^N$ for $\Omega=(0,\ell)$ on $L^2(0,\ell)$. Determine the eigenvalues and the asymptotics of $N(\lambda)/\lambda^{1/2}$ for $\lambda\to\infty$.*

## 6.5 Comments

Theorem 6.1.1 was proved by Mercer [Mer09] in 1909. Here we follow the presentation in the textbook on Functional Analysis by D. Werner [Wer97], where also Exercises 6.4.1 and 6.4.2 are taken from.

## 6.5.1 Tauberian theorems

The Tauberian theorem, Theorem 6.2.1, is due to Karamata [Kar31] from 1931. Our proof is taken from [Sim79, Theorem 10.3], where the elegant proof is attributed to M Aizenman. The notion of an Abelian Theorem has its origin in Abel's Continuity Theorem (1826). Let

$$p(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series, where

$$\sum_{n=0}^{\infty} a_n =: b \quad \text{converges. Then} \quad \lim_{x \uparrow 1} p(x) = b.$$

It was Tauber who proved the following converse result in 1897. If

$$\lim_{n\to\infty} n a_n = 0 \quad \text{and} \quad \lim_{x\uparrow 1} p(x) = b$$

exists, then also

$$\sum_{n=0}^{\infty} a_n = b.$$

The **Tauberian condition**

$$\lim_{n\to\infty} |n a_n| = 0 \quad \text{was weakened by Hardy to} \quad \sup |n a_n| < \infty,$$

a case for which the proof is considerably more difficult (see [ABHN01, Theorem 4.2.17]). This result leads to most interesting investigations and to many different versions of Tauberian theorems which establish an asymptotic behaviour of a function supposing a corresponding asymptotic behaviour of its transform (e.g. Laplace or Fourier Transform). We refer to [ABHN01, Chapter 4] for a class of Tauberian theorems which are particularly interesting in order to determine the asymptotic behaviour as $t \to \infty$ for solutions of evolution equations.

## 6.5.2 Weyl's theorem: Arbitrary domains

In the text, we started from the assumption that $\Omega$ be Dirichlet regular. As we saw, this is equivalent to the fact that the kernel $k^{\Omega}(t,\cdot,\cdot)$ is continuous up to the boundary taking the value 0 on the boundary. From this, we may extend the result to arbitrary bounded domains by the following argument.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Then there exist open sets $\Omega_k \subset \Omega$ which are of class $C^\infty$ such that

$$\Omega_k \subset \Omega_{k+1} \subset \overline{\Omega}_{k+1} \subset \Omega.$$

It follows from Daners [Dan05, Theorem 4.4] that

$$R(\lambda, \Delta^D_{\Omega_k}) \to R(\lambda, \Delta^D_{\Omega}) \quad \text{as } k \to \infty$$

in $\mathcal{L}(L^2(\Omega))$ for all $\lambda \geq 0$. This in turn by [Kat66, IV, §5.3] implies the following.

Let $\lambda \in \sigma(\Delta^D_\Omega)$ and let $\delta > 0$ such that

$$(\lambda - \delta, \lambda + \delta) \cap \sigma(-\Delta^D_\Omega) = \{\lambda\}.$$

Denote by $m$ the multiplicity of $\lambda$. Then there exists $k_0$ such that for $k \geq k_0$, there are exactly $m$ eigenvalues $\lambda_1^k, \ldots, \lambda_m^k$ in $(\lambda - \delta, \lambda + \delta)$ and $\lim_{k\to\infty} \lambda_j^k = \lambda$ $(j = 1, \ldots, m)$. As a consequence, we obtain the following.

Write the eigenvalues of $-\Delta^D_{\Omega_k}$ in the ordering

$$0 < \lambda_1^k \le \lambda_2^k \le \lambda_3^k \le \cdots$$

and similarly the eigenvalues of $-\Delta^D_\Omega$

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

It follows from Courant's Minimax Principle

$$\lambda_j^k = \inf_{\substack{F \subset H_0^1(\Omega_k) \\ \dim F = j}} \sup\{(\nabla u \,|\, \nabla u) : u \in F,\ \|u\|_{L^2} = 1\}, \tag{6.26}$$

that $\lambda_j^{k+1} \le \lambda_j^k$. From Kato's result mentioned above, one sees that

$$\lambda_j = \lim_{k\to\infty} \lambda_j^k.$$

Since $\Omega_k$ is Dirichlet regular (see Examples 5.2.2 d)), we know from (6.24) that

$$\lim_{t\downarrow 0} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j^k t} = \frac{|\Omega_k|}{(4\pi)^{n/2}}.$$

Hence by (6.26),

$$\varliminf_{t\downarrow 0} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j t} \ge \varliminf_{t\downarrow 0} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j^k t} = \frac{|\Omega_k|}{(4\pi)^{n/2}}.$$

for all $k \in \mathbb{N}$. Since $|\Omega_k| \uparrow |\Omega|$ by the Monotone Convergence Theorem, we deduce that

$$\varliminf_{t\downarrow 0} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j t} \ge \frac{|\Omega|}{(4\pi)^{n/2}}. \tag{6.27}$$

Conversely, by (6.20) and (6.21), we have for $t > 0$,

$$t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j^k t} \le \frac{|\Omega_k|}{(2\pi)^{n/2}}.$$

In the spirit of (6.26), it follows from the Monotone Convergence Theorem that

$$t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j t} = \sup_{k\in\mathbb{N}} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j^k t} \le \sup_{k\in\mathbb{N}} \frac{|\Omega_k|}{(2\pi)^{n/2}} = \frac{|\Omega|}{(2\pi)^{n/2}}. \tag{6.28}$$

We have shown by (6.27), (6.28) that

$$\lim_{t\downarrow 0} t^{n/2} \sum_{j=1}^\infty e^{-\lambda_j t} = \frac{|\Omega|}{(4\pi)^{n/2}}.$$

As in Theorem 6.3.3, this implies (6.25) by Karamata's Tauberian Theorem.

### 6.5.3 Weyl's Theorem, references

The proof of Weyl's Theorem we give here is essentially the one given by Kac [Kac66], who found the trace formula (6.23) and used Karamata's Theorem. Here we follow partly Dodzink [Dod81] for Proposition 6.3.1 and 6.3.2, though with a simplified proof. Dodzink assumes that $\Omega$ is of class $C^2$ throughout.

Simon [Sim79] gives a probabilistic proof of Weyl's Theorem assuming that $\partial\Omega$ has Lebesgue measure 0. The argument for arbitrary bounded open sets given in 6.5.2 above might be new.

In contrast to the proofs described so far, which all use the heat kernel, the classical proof by Weyl [Wey11] uses the Courant's Minimax Principle (see (6.26)) and exhausts $\Omega$ by cubes from the interior and approximates $\Omega$ also by cubes from the exterior. Weyl proved the result in a short version in 1911 [Wey11] and in a more extended form in 1912 [Wey12]. This proof can be also found in the textbooks [RS78], [EE87], [CH93]. We refer also to the interesting diploma dissertation of E. Michel [Mic01], where both approaches are presented.

# Lecture 7

# From Forms to Semigroups

The aim of this lecture is to introduce sesquilinear forms. They form an almost algebraic tool to prove that a large class of operators generates holomorphic semigroups on a Hilbert space. We will later consider elliptic differential operators with diverse boundary conditions as a main example. There are three sections.



## 7.1 Coercive forms

Let $V$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. A **sesquilinear form** $a : V \times V \to \mathbb{K}$ is a mapping satisfying

$$\begin{aligned} a(u+v,w) &= a(u,w)+a(v,w) \\ a(\lambda u,w) &= \lambda a(u,w) \\ a(u,v+w) &= a(u,v)+a(u,w) \\ a(u,\lambda v) &= \bar{\lambda} a(u,v) \end{aligned}$$

for $u, v, w \in V, \lambda \in \mathbb{K}$. In other words, $a$ is linear in the first and antilinear in the second variable. If $\mathbb{K} = \mathbb{R}$, then we also say that $a$ is **bilinear**. We frequently say simply **form** instead of sesquilinear/bilinear form. The form $a$ is called **continuous** if there exists $M \geq 0$ such that

$$|a(u,v)| \leq M \|u\|_V \|v\|_V \qquad (u, v \in V). \tag{7.1}$$

Finally, the form $a$ is called **coercive** if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u,u) \geq \alpha \|u\|_V^2 \qquad (u \in V). \tag{7.2}$$

If $\mathbb{K} = \mathbb{R}$, then $V'$ denotes the dual space of $V$. In the case $\mathbb{K} = \mathbb{C}$, we consider antilinear functionals instead of linear functionals. A mapping $f : V \to \mathbb{C}$ is called **antilinear** if

$$f(u+v) = f(u) + f(v) \quad \text{and} \quad f(\lambda u) = \bar{\lambda} f(u) \qquad (u, v \in V,\ \lambda \in \mathbb{C}).$$

The space $V'$ of all continuous antilinear forms is a complex Banach space for the norm

$$\|f\| = \sup_{\|u\|_V \leq 1} |f(u)|.$$

We call it the **antidual** of $V$. Frequently we write

$$\langle f, u \rangle = f(u) \quad (u \in V,\ f \in V').$$

Now we continue to treat simultaneously the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. A form $a : V \times V \to \mathbb{K}$ is called **symmetric** if

$$a(u,v) = \overline{a(v,u)}.$$

Thus, a continuous, coercive, symmetric form on $V$ is the same as a scalar product on $V$ that induces an equivalent norm (we say an **equivalent scalar product**). So the following theorem is a generalisation of the Theorem of Riesz–Fréchet to a non-symmetric form.

**Theorem 7.1.1** (Lax–Milgram). *Let $a : V \times V \to \mathbb{K}$ be a continuous, coercive form. Then there exists an isomorphism $\mathcal{A} : V \to V'$ such that*

$$\langle \mathcal{A}u, v \rangle = a(u,v) \tag{7.3}$$

*for all $u \in H$, $v \in V$. Moreover, $\|\mathcal{A}^{-1}\|_{\mathcal{L}(V',V)} \leq \frac{1}{\alpha}$, where $\alpha$ is the constant of* (7.2).

*Proof.* By the Theorem of Riesz–Fréchet, for each $u \in V$ there exists a unique $Bu \in V$ such that

$$(Bu \mid v)_V = a(u,v) \qquad (v \in V).$$

By the continuity assumption (7.1), $\|Bu\|_V \leq M\|u\|_V$. It follows from the coerciveness assumption (7.2) that $\alpha\|u\|_V^2 \leq \operatorname{Re}\ a(u,u) \leq \|Bu\|_V \|u\|_V$, hence

$$\alpha \|u\|_V \leq \|Bu\|_V \qquad (u \in V). \tag{7.4}$$

This implies that $B$ is injective and its range $R(B)$ is closed. We show that $R(B) = V$. For this it suffices to show that $R(B)$ is dense in $V$, i.e., that $R(B)^\perp = \{0\}$ in $V$. Let $v \in R(B)^\perp$. Then $0 = (Bu \mid v)_V = a(u,v)$ for all $u \in V$. In particular, $a(v,v) = 0$, hence $v = 0$ by (7.2). We have shown that $B$ is bijective. It follows from (7.4) that $\|B^{-1}\|_{\mathcal{L}(V)} \leq \frac{1}{\alpha}$. Consider the mapping $V \ni x \mapsto f_x \in V'$, where

$$f_x(y) = (x \mid y)_V \qquad (y \in V).$$

Since by the Theorem of Riesz–Fréchet this mapping is an isometric isomorphism, the claim follows. $\square$

The space $V'$ is always isomorphic to $V$ (and thus a Hilbert space). To see this, we may apply Theorem 7.1.1 to the usual scalar product. But for the applications we have in mind another identification of $V'$ will be more useful. This identification depends on an additional Hilbert space.

In fact, now we assume that the Hilbert space $V$ is *continuously and densely injected into another Hilbert space* $H$, i.e., $V \subset H$ and there exists a constant $c > 0$ such that

$$\|u\|_H \leq c\|u\|_V \qquad (u \in V)$$

and $V$ is dense in $H$ for the norm of $H$. We express this by

$$V \overset{d}{\hookrightarrow} H.$$

Now we may inject $H$ continuously into $V'$ using the scalar product of $H$ in the following way: for $u \in H$ let $j(u) \in V'$ be given by

$$\langle j(u), v\rangle = (u \mid v)_H \qquad (v \in V).$$

Then

$$\begin{aligned} \|j(u)\|_{V'} &= \sup_{\|v\|_V \leq 1} |(u \mid v)_H| \leq \sup_{\|v\|_V \leq 1} \|u\|_H \|v\|_H \\ &\leq c\|u\|_H. \end{aligned}$$

Thus $j$ is a continuous linear mapping. Moreover, $j$ is injective. In fact, if $j(u) = 0$, then $\|u\|_H^2 = (u \mid u)_H = \langle j(u), u\rangle = 0$. Hence $u = 0$.

In the following we identify $H$ with a subspace of $V'$ omitting the identification mapping $j$, i.e., we write

$$\langle u, v\rangle = (u \mid v)_H$$

for all $u, v \in V$ where $\langle u, v\rangle = \langle j(u), v\rangle$.

We emphasise that the identification of $V$ with a subspace of $V'$ depends crucially on the choice of the Hilbert space $H$. Resuming the continuous injections defined above, we have

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'.$$

Lemma 7.1.3 together with Theorem 7.1.4 below show in particular that the space $V$ is dense in $V'$, thus also $H$ is dense in $V'$ as we indicated above.

The following example illustrates the identifications made above. We will see in Lecture 8 that it describes already the most general situation, up to unitary equivalence.

**Example 7.1.2.** *Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\alpha > 0$, and $m : \Omega \to [\alpha, \infty)$ be measurable. Let $H = L^2(\Omega, \mu)$, $V = L^2(\Omega, m\mu)$. Then $V \hookrightarrow H$. We have $V' = L^2(\Omega, \frac{1}{m}\mu)$ if we write the duality as*

$$\langle f, u \rangle = \int_\Omega f\bar{u}d\mu \qquad (u \in V,\ f \in L^2(\Omega, \frac{1}{m}\mu)).$$

Before proving the first generation result we need an auxiliary result.

**Lemma 7.1.3.** *Let $A$ be an operator on a reflexive space $X$ such that $[\omega, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \le M$ for $\lambda \ge \omega$. Then $D(A)$ is dense in $X$.*

*Proof.* Let $x \in X$. Since $X$ is reflexive there exists $\lambda_k \to \infty$ and $y \in X$ such that $x_k := \lambda_k R(\lambda_k, A)x \rightharpoonup y$ as $k \to \infty$. Let $\mu \in \rho(A)$. It follows that $R(\mu, A)x_k \rightharpoonup R(\mu, A)y$. But by the Resolvent Identity

$$R(\mu, A)x_k = \frac{\lambda_k}{\lambda_k - \mu}(R(\mu, A)x - R(\lambda_k, A)x) \rightharpoonup R(\mu, A)x.$$

Thus $R(\mu, A)y = R(\mu, A)x$, which implies $x = y$. We have shown that $x$ is in the weak closure of $D(A)$, which coincides with the strong closure by the Theorem of Hahn–Banach. □

Now we will associate three holomorphic $C_0$-semigroups with the form $a$, namely on the spaces $V'$, $H$, and $V$. The most important one is the semigroup on $H$. But we start by considering $V'$. For this case we merely consider $\mathbb{K} = \mathbb{C}$ for simplicity. Let $a : V \times V \to \mathbb{C}$ be a continuous, coercive form and denote by $\mathcal{A} : V \to V'$ the associated operator defined as in (7.3). We may see $\mathcal{A}$ as an unbounded closed operator on the Banach space $V'$. Here is our first generation result.

**Theorem 7.1.4.** *The operator $-\mathcal{A}$ generates a bounded holomorphic semigroup on $V'$.*

*Proof.* For $\operatorname{Re}\lambda \ge 0$ we consider the form $a_\lambda$ defined by $a_\lambda(u, v) = \lambda(u \mid v)_H + a(u, v)$. Then $a_\lambda$ is continuous and coercive and the associated operator is $\lambda + \mathcal{A}$. Thus $\lambda + \mathcal{A} : V \to V'$ is an isomorphism and $\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(V',V)} \le \frac{1}{\alpha}$ for all $\operatorname{Re}\lambda \ge 0$ by Theorem 7.1.1. Since $\lambda(\lambda + \mathcal{A})^{-1} + \mathcal{A}(\lambda + \mathcal{A})^{-1} = I$ it follows that

$$\begin{aligned}
\|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V')} &\le 1 + \|\mathcal{A}(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V')} \\
&\le 1 + \|\mathcal{A}\|_{\mathcal{L}(V,V')}\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V',V)} \\
&\le 1 + \frac{1}{\alpha}\|\mathcal{A}\|_{\mathcal{L}(V,V')}
\end{aligned}$$

for $\operatorname{Re}\lambda \ge 0$. This proves the holomorphic estimate of Theorem 2.5.3. Since $\mathcal{A}$ is densely defined, by the preceding lemma, the proof is complete. □

Now we consider again $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ simultaneously. Let $a : V \times V \to \mathbb{K}$ be a continuous, coercive form where $V \overset{d}{\hookrightarrow} H$. We define an operator $A$ on $H$ by

$$\begin{aligned} D(A) &:= \{u \in V : \exists f \in H \; a(u,v) = (f \mid v)_H \text{ for all } v \in V\} \\ Au &:= f. \end{aligned}$$

Note that $f$ is uniquely determined by $u$ since $V$ is dense in $H$. We call $A$ the **operator associated with** $a$ on $H$. Now we can prove the main generation theorem.

**Theorem 7.1.5.** *The operator $-A$ generates a holomorphic semigroup on $H$.*

*Proof.*

1. Let first $\mathbb{K} = \mathbb{C}$. Consider the operator $\mathcal{A}$ on $V'$ defined in Theorem 7.1.1. Since also the form $a_\lambda$ given by $a_\lambda(u,v) = a(u,v) + \lambda(u \mid v)_H$ is coercive for $\operatorname{Re}\lambda \geq 0$, the operator $\mathcal{A} + \lambda$ is invertible for all $\operatorname{Re}\lambda \geq 0$. Observe that $D(\mathcal{A}) = V \hookrightarrow H$. Hence $R(\lambda, \mathcal{A})H \subset H$. Consequently $\rho(\mathcal{A}) \subset \rho(A)$ and $R(\lambda, A) = R(\lambda, \mathcal{A})_{|H}$ for all $\lambda \in \rho(\mathcal{A})$.

   In particular for $\operatorname{Re}\lambda \geq 0$. Let $f \in H$ and $u = (\lambda + A)^{-1} f$ where $\operatorname{Re}\lambda \geq 0$. Then $u \in V$ and
   $$\lambda(u \mid v)_H + a(u \mid v) = (f \mid v)_H \qquad (v \in V).$$
   In particular,
   $$\lambda\|u\|_H^2 + a(u,u) = (f \mid u)_H.$$
   This implies
   $$\alpha\|u\|_V^2 \leq \operatorname{Re} a(u,u) = \operatorname{Re}(f \mid u)_H - \operatorname{Re}\lambda\|u\|_H^2 \leq \|f\|_H\|u\|_H$$
   and
   $$|\lambda|\|u\|_H^2 \leq \|f\|_H\|u\|_H + M\|u\|_V^2.$$
   Hence,
   $$|\lambda|\|u\|_H^2 \leq \left(\frac{M}{\alpha} + 1\right)\|f\|_H\|u\|_H$$
   and so $|\lambda|\|u\|_H \leq \left(\frac{M}{\alpha} + 1\right)\|f\|_H$. We have proved that
   $$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(H)} \leq \left(\frac{M}{\alpha} + 1\right) \qquad (\operatorname{Re}\lambda \geq 0).$$

   Since by Lemma 7.1.3 the operator $A$ is densely defined, it follows from Theorem 2.5.3 that $-A$ generates a holomorphic $C_0$-semigroup.

2. Assume now that $\mathbb{K} = \mathbb{R}$. Then we can extend the bilinear form $a$ to a unique sesquilinear form $a_{\mathbb{C}}$ on $V_{\mathbb{C}}$. It is not difficult to see that $a_{\mathbb{C}}$ is continuous and coercive. The operator $A_{\mathbb{C}}$ on $H_{\mathbb{C}}$ associated with $a_{\mathbb{C}}$ generates a holomorphic $C_0$-semigroup $(T_{\mathbb{C}}(t))_{t\geq 0}$. Since $R(\lambda, A_{\mathbb{C}})H \subset H$ for $\lambda \geq 0$, this semigroup leaves $H$ invariant by Euler's Formula (2.15). Its restriction $(T(t))_{t\geq 0}$ to $H$ is a $C_0$-semigroup whose generator is $-A$.

□

Finally, we want to show that also a holomorphic $C_0$-semigroup on $V$ is induced by the coercive form $a$. For this we use the following general result.

**Proposition 7.1.6.** *Let $B$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Consider $D(B)$ as a Banach space endowed with the graph norm. Then $(T_1(t))_{t\geq 0} := (T(t)_{|D(B)})_{t\geq 0}$ is a $C_0$-semigroup on $D(B)$ whose generator is the operator $B_1$ given by*

$$B_1 x := Bx \qquad (x \in D(B_1) := D(B^2)).$$

*Proof.* We may assume that $0 \in \rho(B)$ (considering $B - \omega$ instead of $B$ otherwise, $\omega \in \rho(B)$). Then $B$ is an isomorphism from $D(B)$ onto $X$. Hence $T_1(t) := B^{-1}T(t)B$ defines a $C_0$-semigroup on $D(B)$. Since $T(t)Bx = BT(t)x$ for $x \in D(B)$, $T_1(t)x = T(t)x$ for $x \in D(B)$. It is easy to see that the generator of $(T_1(t))_{t\geq 0}$ is $B_1$. □

From Proposition 7.1.6 we obtain directly the following.

**Proposition 7.1.7.** *The semigroup $(e^{-tA})_{t\geq 0}$ on $H$ leaves $V$ invariant and its restriction is a holomorphic $C_0$-semigroup on $V$.*

*Proof.* By the proof of Proposition 7.1.6 the semigroup $(e^{-tA}_{|V})_{t\geq 0}$ is similar to the semigroup generated by $-\mathcal{A}$ on $V'$. From this it follows that also $(e^{-tA}_{|V})_{t\geq 0}$ is holomorphic. □

**Corollary 7.1.8.** *The domain $D(A)$ of $A$ is dense in $V$.*

*Proof.* Let $u \in V$. Then $\lim_{t\to 0+} e^{-tA}u = u$ in $V$ by Proposition 7.1.7. Since $e^{-tA}u \in D(A)$ for all $t > 0$, the claim follows. □

In conclusion, we obtained three $C_0$-semigroups associated with $a$. The semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$ on $V'$ leaves invariant $H$ and $V$. The restriction are holomorphic $C_0$-semigroups with generators $-A$ on $H$ and $-A_V$ on $V$.

For the applications one will proceed as follows: Given the Hilbert space $H$ and an operator $A$ on $H$, one will try to find $V \overset{d}{\hookrightarrow} H$ and a form $a : V \times V \to \mathbb{K}$ such that $A$ is associated with $a$.

We conclude this section with a typical example of a coercive form.

**Example 7.1.9** (strictly elliptic operators of pure second order)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $H = L^2(\Omega)$. Let $a_{ij} \in L^\infty(\Omega)$, $1 \leq i, j \leq n$, such that*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for a.e. $x \in \Omega$. Let $V = H_0^1(\Omega)$ and define $a : V \times V \to \mathbb{K}$ by*

$$a(u, v) := \int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_iu(x)D_jv(x)dx.$$

*Then $a$ is continuous. In fact*

$$|a(u, v)| \leq c\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq c\|u\|_{H^1}\|v\|_{H^1},$$

*where $c = \| \sum_{i,j=1}^n |a_{ij}| \ \|_{L^\infty}$.*

*Moreover,*

$$\operatorname{Re} a(u, u) \geq \alpha \int_\Omega |\nabla u|^2 dx \qquad (u \in H_0^1(\Omega)).$$

*It follows from Poincaré's Inequality that $a$ is coercive.*

*Let $A$ be the operator on $L^2(\Omega)$ associated with $a$. Then $-A$ generates a holomorphic $C_0$-semigroup on $L^2(\Omega)$. If the coefficients are merely measurable, then one can describe the operator $A$ not much better than by the form. But the use of distributions makes it more elegant.*

*If $u \in H_0^1(\Omega)$, then $D_ju \in L^2(\Omega)$ and hence $a_{ij}D_ju \in L^2(\Omega) \subset \mathcal{D}(\Omega)'$. Then also $D_ia_{ij}D_ju \in \mathcal{D}(\Omega)'$. Define the operator $B : H_0^1(\Omega) \to \mathcal{D}(\Omega)'$ by*

$$Bu := \sum_{i,j=1}^n D_ia_{ij}D_ju \qquad (u \in H_0^1(\Omega)).$$

*Then*

$$D(A) = \left\{u \in H_0^1(\Omega) : Bu \in L^2(\Omega)\right\}$$

*and*

$$-Au = Bu \qquad (u \in D(A)).$$

*This means that $-A$ is the* part of $B$ *in $L^2(\Omega)$.*

## 7.2 Elliptic forms and rescaling

If $A$ generates the $C_0$-semigroup $(T(t))_{t\geq 0}$, then $A + \omega I$ generates the $C_0$-semigroup $(e^{\omega t}T(t))_{t\geq 0}$. This observation allows us to pass from the class of coercive forms, which were the subject of Section 7.1, to elliptic forms.

Let $V, H$ be Hilbert spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ such that $V \overset{d}{\hookrightarrow} H$. Let $a : V \times V \to \mathbb{K}$ be a continuous sesquilinear form. We call $a$ **elliptic** (or more precisely $H$**-elliptic**) if

$$\operatorname{Re} a(u,u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \qquad (u \in V) \tag{7.5}$$

for some $\omega \in \mathbb{R}$, $\alpha > 0$. This is equivalent to saying that the form $a_\omega : V \times V \to \mathbb{K}$ defined by

$$a_\omega(u,v) := a(u,v) + \omega(u \mid v)_H \qquad (u, v \in V)$$

is coercive.

We define the **operator** $A$ **on** $H$ **associated with** $a$ by

$$\begin{array}{rcl} D(A) & := & \{u \in V : \exists f \in H \text{ s.t. } a(u,v) = (f \mid v)_H \text{ for all } v \in V\}, \\ Au & := & f. \end{array} \tag{7.6}$$

Note that $A$ is well-defined since we assume that $V$ is dense in $H$. It is clear that the operator $A + \omega$ is associated with the coercive form $a_\omega$. Hence $-(A+\omega)$ generates a holomorphic $C_0$-semigroup $T_\omega$ on $H$. Consequently, $-A$ generates the semigroup $T$ given by $T(t) = e^{\omega t} T_\omega(t)$ $(t \geq 0)$. Then $(T(t))_{t \geq 0}$ is called the **semigroup associated with** $a$.

The following easy perturbation result will give us a first example of an elliptic form which is not coercive.

**Proposition 7.2.1** (perturbation)**.** *Let* $B \in \mathcal{L}(V, H)$ *and* $a$ *be coercive. Then the form* $b : V \times V \to \mathbb{K}$ *given by*

$$b(u,v) := a(u,v) + (Bu \mid v)_H \qquad (u, v \in V)$$

*is continuous and elliptic.*

*Proof.* One has

$$|b(u,v)| \leq M\|u\|_V \|v\|_V + \|B\| \|u\|_V \|v\|_H \leq (M + c\|B\|) \|u\|_V \|v\|_V,$$

where $c$ is such that $\|\cdot\|_H \leq c\,\|\cdot\|_V$. Moreover, for $u \in V$

$$\begin{array}{rcl} \operatorname{Re} b(u,u) & \geq & \operatorname{Re} a(u,u) - \|B\| \|u\|_V \|v\|_H \\ & \geq & \alpha \|u\|_V^2 - \|B\| \|u\|_V \|v\|_H \\ & = & \alpha \|u\|_V^2 - \|B\| \varepsilon \|u\|_V \dfrac{1}{\varepsilon} \|v\|_H \\ & \geq & \alpha \|u\|_V^2 - \varepsilon^2 \|u\|_V^2 - \|B\|^2 \dfrac{1}{\varepsilon^2} \|v\|_H^2, \end{array}$$

where we need the inequality $ab \leq a^2 + b^2$. Hence $\operatorname{Re} b(u,u) + \|B\|^2 \frac{1}{\varepsilon^2} \|v\|_H^2 \geq (\alpha - \varepsilon^2) \|u\|_V^2$. Taking $\varepsilon^2 < \alpha$ shows that the form is elliptic. $\square$

Now we can perturb elliptic operators by lower order coefficients.

**Example 7.2.2** (elliptic operators). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $a_{ij} \in L^\infty(\Omega)$, $1 \le i, j \le n$, such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for a.e. $x \in \Omega$. Let $b_i \in L^\infty(\Omega)$, $0 \le i \le n$. Then*

$$a(u,v) := \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)D_iu(x)D_jv(x)dx + \sum_{i=1}^{n}\int_\Omega b_iD_iuvdx + \int_\Omega b_0uvdx$$

*defines a continuous form on $H_0^1(\Omega) \times H_0^1(\Omega)$ which is $L^2(\Omega)$-elliptic. In fact, it suffices to let $B : H_0^1(\Omega) \to L^2(\Omega)$ be given by*

$$Bu = \sum_{i=1}^{n} b_iD_iu + b_0u \qquad (u \in H_0^1(\Omega))$$

*and apply Proposition 7.2.1.*

## 7.3 Contractivity properties

In the following we assume throughout that $a$ is elliptic. We now establish several properties of $(T(t))_{t\ge 0}$ keeping the assumptions made above. First we consider contractivity. We need the following general result.

**Proposition 7.3.1.** *Let $B$ be the generator of a $C_0$-semigroup $(S(t))_{t\ge 0}$ on $H$. Then $\|S(t)\| \le 1$ for all $t \ge 0$ if and only if $B$ is dissipative.*

*Proof.* Assume that $B$ is dissipative, i.e.,

$$\mathrm{Re}(Bu \mid u) \le 0 \qquad (u \in D(B)).$$

Let $u \in D(B)$. Then

$$\begin{aligned}\frac{d}{dt}\|T(t)u\|_H^2 &= \frac{d}{dt}(T(t)u \mid T(t)u)_H = (BT(t)u \mid T(t)u)_H + (T(t)u \mid BT(t)u)_H \\ &= 2\,\mathrm{Re}(BT(t)u \mid T(t)u)_H \le 0.\end{aligned}$$

It follows that $\|T(t)u\|_H^2$ is decreasing. In particular, $\|T(t)u\|_H \le \|u\|_H$ for all $t \ge 0$. Since $D(B)$ is dense in $H$, the claim follows.

Conversely, assume that $T$ is contractive. Let $u \in D(B)$. Then

$$\|T(t+s)u\|_H = \|T(t)T(s)u\|_H \le \|T(s)u\|_H \qquad (t, s \ge 0).$$

Hence $\|T(\cdot)u\|_H^2$ is decreasing and it follows that

$$\operatorname{Re}(Bu \mid u)_H = \frac{d}{dt}_{|t=0} \|T(t)u\|_H^2 \leq 0.$$

□

We say that the sesquilinear form $a$ is **accretive** if

$$\operatorname{Re} a(u,u) \geq 0 \qquad (u \in V).$$

**Proposition 7.3.2.** *Consider the semigroup $(T(t))_{t\geq 0}$ on $H$ associated with $a$. Then $(T(t))_{t\geq 0}$ is contractive if and only if $a$ is accretive.*

*Proof.* If $a$ is accretive, then $\operatorname{Re}(Au \mid u)_H = \operatorname{Re} a(u,u) \geq 0$ for all $u \in D(A)$. Thus $-A$ is dissipative and the semigroup is contractive by Proposition 7.3.1.

Conversely, assume $(T(t))_{t\geq 0}$ to be contractive. Then $-A$ is dissipative, hence

$$\operatorname{Re} a(u,u) = \operatorname{Re}(Au \mid u)_H \geq 0 \qquad (u \in D(A)).$$

Since $D(A)$ is dense in $V$ by Corollary 7.1.8, it follows that $\operatorname{Re} a(u,u) \geq 0$ for all $u \in V$. □

**Example 7.3.3.** *Let $\Omega \subset \mathbb{R}^n$ be open, $H = L^2(\Omega)$. Let $a_{ij} \in L^\infty(\Omega)$, $1 \leq i,j \leq n$, such that*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for a.e. $x \in \Omega$. Let $V = H_0^1(\Omega)$ and define $a : V \times V \to \mathbb{K}$ by*

$$a(u,v) := \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j u(x) dx \qquad (u \in H_0^1(\Omega)).$$

*Then $a : V \times V \to \mathbb{K}$ is continuous and $H$-elliptic. Moreover, $a$ is accretive. Let $A$ be the associated operator on $L^2(\Omega)$. Then $-A$ generates a contractive $C_0$-semigroup on $L^2(\Omega)$. In general the form $a$ is not coercive if $\Omega$ is unbounded.*

Next we establish an asymptotic property of $(T(t))_{t\geq 0}$ as $t \to 0+$. Recall that $(T(t))_{t\geq 0}$ is holomorphic, hence for all $t > 0$ the operator $T(t)$ is bounded from $H$ into $D(A)$, where $D(A)$ carries the graph norm. Since $D(A) \subset V$ and $V \hookrightarrow H$ it follows from the Closed Graph Theorem that

$$D(A) \hookrightarrow V. \tag{7.7}$$

Consequently, $T(t) \in \mathcal{L}(H,V)$.

**Proposition 7.3.4.** *There exists a constant $c > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(H,V)} \le ct^{-\frac{1}{2}} \qquad (0 < t \le 1). \tag{7.8}$$

*Proof.* Considering the form $a_\omega$ instead of $a$ we may assume that $a$ is coercive. Then $a$ is accretive and

$$\|T(t)\|_{\mathcal{L}(H)} \le 1 \qquad (t \ge 0).$$

Since $(T(t))_{t\ge 0}$ is holomorphic, by Definition 2.5.1 there is a constant $c > 0$ such that

$$\|AT(t)u\|_H \le \frac{c}{t}\|u\|_H \qquad (0 < t \le 1,\ u \in H).$$

Hence, by coercivity, one has for all $u \in H$

$$\begin{aligned} \|T(t)u\|_V^2 \le \alpha \operatorname{Re} a(T(t)u, T(t)u) &= \alpha \operatorname{Re}(AT(t)u \mid T(t)u)_H \\ &\le \alpha \|AT(t)u\|_H \|T(t)u\|_H \\ &\le \frac{\alpha c}{t}\|u\|_H^2. \end{aligned}$$

This implies (7.8). □

## 7.4 Exercises

In the first exercise exponential stability is established for coercive forms.

**Exercise 7.4.1** (exponential stability)**.** *Let $V \overset{d}{\hookrightarrow} H$ and let $a : V \times V \to \mathbb{C}$ be continuous and coercive. Denote by $A$ the associated operator. Show that*

$$\|e^{-tA}\|_{\mathcal{L}(H)} \le e^{-\epsilon t} \qquad (t \ge 0)$$

*for some $\epsilon > 0$. Give a concrete example.*

**Exercise 7.4.2** (an elliptic operator)**.** *Let $b \in L^\infty(0,1)$. Define the operator $A$ on $L^2(0,1)$ by*

$$Au = u'' + bu' \qquad (u \in D(A) := \{u \in H^2(0,1) : u'(0) = u'(1) = 0\}).$$

*Show that $A$ generates a holomorphic $C_0$-semigroup.*

> **Hint**: Define an appropriate form. Recall that $H^2(0,1) \subset C^1[0,1]$, cf. Theorem 3.5.3 and Exercise 3.5.4.

**Exercise 7.4.3.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\ge 0}$ on a Hilbert space $H$. Assume that $V$ is a second Hilbert space such that $D(A) \subset V \subset H$. Assume that $\operatorname{Re}(Au \mid u)_V \le 0$ for all $u \in D(A)$ such that $Au \in V$.*

*a) Consider the part $A_V$ of $A$ in $V$, i.e.,*

$$\begin{aligned} D(A_V) &:= \{u \in D(A) : Au \in V\}, \\ A_V u &:= Au. \end{aligned}$$

*Show that $A_V$ generates a $C_0$-semigroup $(S(t))_{t\ge 0}$ on $V$.*

*b) Show that $S(t) = T(t)_{|V}$ $(t \ge 0)$*

*c) Deduce Theorem 7.1.5 from Theorem 7.1.4, except for the property of holomorphy.*

**Exercise 7.4.4.** *Give a detailed proof of the assertions of Example 7.1.2.*

## 7.5 Comments

There are two different but equivalent approaches to forms. The one we chose here starts by a given Hilbert space $V$ as form domain. This approach can be found in the monographs [DL88] and [Tan79].

Another approach consists in considering a sesquilinear form $a$ on a Hilbert space with a domain $D(a)$ which is just a subspace of $H$ but does not carry any further structure. Then it becomes a unitary space by a scalar product defined by means of the form. We will describe this approach leading to **closed forms** in Lecture 8.

But we will keep our presentation from Lecture 7 in the sequel of the course.

# Lecture 8

# More on forms

In this lecture we continue to talk about form methods. At first we show that all $m$-dissipative symmetric operators are associated with a form.

Then we describe a different but equivalent way to present forms. We conclude the section by two easy perturbation results. There are three sections.

- Symmetric forms
- Closed forms
- Form sums and multiplicative perturbations

## 8.1 Symmetric forms

We start by a remark on adjoints and selfadjointness. Let $H$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $A$ be a densely defined operator on $H$ with domain $D(A)$. Then the **adjoint** $A^*$ of $A$ is defined by

$$\begin{aligned} D(A^*) &:= \{u \in H : \exists f \in H \text{ s.t. } (Av \mid u)_H = (v \mid f)_H \text{ for all } v \in D(A)\}, \\ A^*u &:= f. \end{aligned}$$

Since $D(A)$ is dense in $H$ the element $f$ is uniquely determined by $u$.

**Proposition 8.1.1.** *Assume that $\lambda \in \rho(A) \cap \mathbb{R}$.*

*a) Then $\lambda \in \rho(A^*)$ and $R(\lambda, A)^* = R(\lambda, A^*)$.*

*b) The following are equivalent:*

*i) $A = A^*$,*

*ii) $A$ is symmetric,*

*iii)* $R(\lambda, A)^* = R(\lambda, A)$.

*If (i) holds, then we say that A is* **selfadjoint**.

The proof is not difficult and can be omitted.

Now we consider a second Hilbert space $V$ over $\mathbb{K}$ such that $V \overset{d}{\hookrightarrow} H$. Let $a : V \times V \to \mathbb{K}$ be a continuous, elliptic sesquilinear form. Let now $A$ be the operator associated with $a$. So clearly $\rho(A) \cap \mathbb{R} \neq \emptyset$ and we can apply Proposition 8.1.1. Let $(T(t))_{t\geq 0}$ be the $C_0$-semigroup generated by $-A$. Denote by $a^* : V \times V \to \mathbb{K}$ the **adjoint form** of $a$, which is given by

$$a^*(u, v) := \overline{a(v, u)} \qquad (u, v \in V).$$

**Proposition 8.1.2.** *The adjoint $A^*$ of $A$ coincides with the operator on $H$ that is associated with $a^*$. Moreover, $-A^*$ generates the adjoint semigroup $(T(t)^*)_{t\geq 0}$ of $(T(t))_{t\geq 0}$.*

*Proof.* Replacing $a$ by $a + \omega(\cdot \mid \cdot)_H$ if necessary, we may assume that $a$ is coercive. Then also $a^*$ is coercive. Let $B$ be the operator associated with $a^*$. Let $u \in D(A)$, $w \in D(B)$. Then

$$(Au \mid w)_H = a(u, w) = \overline{a^*(w, u)} = \overline{(Bw \mid u)_H} = (u \mid Bw)_H.$$

This shows that $B \subset A^*$. Since $0 \in \rho(B) \cap \rho(A^*)$ it follows that $B = A^*$. By Proposition 8.1.1(a) and Euler's formula (2.15) it follows that the $C_0$-semigroup generated by $-A^*$ is $(T(t)^*)_{t\geq 0}$. □

**Corollary 8.1.3.** *The following assertions are equivalent.*

*i)* $A$ *is selfadjoint,*

*ii)* $a = a^*$,

*iii)* $T(t) = T(t)^*$ $(t \geq 0)$.

We say that the form $a$ is **symmetric** if $a = a^*$.

Next we reconsider multiplication operators as illustrating example.

**Example 8.1.4** (multiplication operators)**.** *Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\alpha > 0$ and $m : \Omega \to [\alpha, \infty)$ a measurable function. Let $H = L^2(\Omega, \mu)$ and $V = L^2(\Omega, m d\mu)$ with $\|f\|_V^2 = \int_\Omega |f|^2 m d\mu$. Then $V$ is a Hilbert space and $V \overset{d}{\hookrightarrow} H$.*

*Let $a : V \times V \to \mathbb{K}$ be given by $a(u, v) := \int_\Omega u\bar{v} m d\mu$. Then $a$ is a coercive, continuous form. The operator $A$ on $H$ associated with $a$ on $H$ is given by*

$$Af := mf \qquad (f \in D(A) = L^2(\Omega, m^2 d\mu)).$$

*The space $V'$ can be identified with $L^2(\Omega, \frac{1}{m}d\mu)$ by letting*

$$\langle f, u\rangle := \int_\Omega f u d\mu \qquad (u \in V = L^2(\Omega, md\mu),\ f \in L^2(\Omega, \frac{1}{m}d\mu)).$$

*Then the operator $\mathcal{A} : V \to V'$ associated with $a$ is given by*

$$\mathcal{A}u = mu \qquad (u \in D(\mathcal{A}) = V = L^2(\Omega, md\mu)).$$

*The $C_0$-semigroup on $L^2(\Omega, \mu)$ generated by $-A$ is given by*

$$e^{-tA}f = e^{-tm}f, \qquad (f \in L^2(\Omega, \mu)),$$

*and also the semigroup generated by $-\mathcal{A}$ on $L^2(\Omega, \frac{1}{m}\mu)$ is given by*

$$e^{-t\mathcal{A}}f = e^{-tm}f \qquad (f \in L^2(\Omega, \frac{1}{m}\mu)).$$

By the Spectral Theorem, any symmetric, coercive, continuous form on a separable Hilbert space is unitarily equivalent to that considered in Example 8.1.4. This is made precise in the exercises.

Finally we reconsider the examples introduced before.

**Example 8.1.5** (the Dirichlet Laplacian revisited)**.** *Let $\mathbb{K} = \mathbb{R}$ for simplicity. Let $\Omega \subset \mathbb{R}^n$ be open, $V = H_0^1(\Omega)$, and let $a : V \times V \to \mathbb{R}$ be given by*

$$a(u, v) = \int_\Omega \nabla u \nabla v dx.$$

*If $\Omega$ is contained in a strip, then $a$ is coercive by Poincaré's inequality. Let $H = L^2(\Omega)$. Then the operator associated with $a$ on $H$ is $-\Delta_\Omega^D$.*

**Example 8.1.6** (the Neumann Laplacian revisited)**.** *Let $\mathbb{K} = \mathbb{R}$ for simplicity. Let $\Omega \subset \mathbb{R}^n$ be open, $V = H^1(\Omega)$, and let $a : V \times V \to \mathbb{R}$ be given by*

$$a(u, v) = \int_\Omega \nabla u \nabla v \, dx.$$

*Let $H = L^2(\Omega)$. Then $a$ is continuous, elliptic and accretive. The operator associated with $a$ on $H$ is $-\Delta_\Omega^N$.*

The form corresponding to the Dirichlet Laplacian is the restriction of the form corresponding to the Neumann Laplacian. Thus, different domains lead in general to different semigroups. This contrasts the corresponding situation for generators: If $A$ and $B$ are two generators of a $C_0$-semigroup on a Banach space $X$ and if $A \subset B$, then $A = B$ and the semigroups coincide. (Here $A \subset B$ means by definitions that $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$.)

It is worth it to consider also diagonal operators in the new framework. They occur after a similarity transformation if a symmetric $m$-dissipative operator has compact resolvent.

**Example 8.1.7** (selfadjoint operator with compact resolvent)**.** *Let $-A$ be symmetric and $m$-dissipative with compact resolvent on a separable Hilbert space. Then up to a unitary equivalence we have $H = \ell^2$, $Ax = (\lambda_n x_n)_{n\in\mathbb{N}}$, and $D(A) = \{x \in \ell^2 : \sum_{n=1}^{\infty} |x_n|^2\lambda_n^2 < \infty$, where $\lambda_n \geq 0$ and $\lim_{n\to\infty} \lambda_n = \infty\}$, see Section 1.4.*

*Let $V = \{x \in \ell^2 : \sum_{n=1}^{\infty} |x_n|^2\lambda_n < \infty\}$, $\|x\|_V = (\sum_{n=1}^{\infty} |x_n|^2(\lambda_n + 1))^{\frac{1}{2}}$. Then $V \overset{d}{\hookrightarrow} H$. Let $a(x, y) = \sum_{n=1}^{\infty} \lambda_n x_n \overline{y_n}$. Then $a$ is continuous on $V \times V$ and elliptic (in fact, $\operatorname{Re} a(x, x) + \|x\|_H^2 = \|x\|_V^2$). Now $A$ is precisely the operator associated with $a$.*

From the above example we can deduce the following.

**Proposition 8.1.8.** *Let $a$ be a symmetric, continuous, elliptic form defined on $V \times V$ where $V \overset{d}{\hookrightarrow} H$. Denote by $A$ the associated operator on $H$. Then $(e^{-tA})_{t\geq0}$ is compact if and only if the injection $V \hookrightarrow H$ is compact.*

*Proof.* Assume $(e^{-tA})_{t\geq0}$ to be compact for $t > 0$. Then $A$ has compact resolvent by Proposition 2.5.7. We may assume that $-A$ is dissipative (replacing $A$ by $A+\omega$ otherwise). Then up to unitary equivalence the form is given as in Example 8.1.7. Since the injection $V \hookrightarrow \ell^2$ is compact, the claim follows.

Conversely, assume that the injection $V \hookrightarrow H$ is compact. It follows from (7.7) that the injection $D(A) \hookrightarrow V \hookrightarrow H$ is compact. Hence $(e^{tA})_{t\geq0}$ is compact for $t > 0$ by Proposition 2.5.7. □

As a corollary we obtain a heat kernel proof of the Rellich-Kondrachov Theorem.

**Corollary 8.1.9.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

*Proof.* The operators $e^{t\Delta_\Omega^D}$ are compact for $t > 0$ by Corollary 4.2.5. The form domain of $\Delta_\Omega^D$ is $H_0^1(\Omega)$ by Example 8.1.5. So the claim follows from Proposition 8.1.8. □

## 8.2 Closed forms

In this section we describe a different but equivalent approach to elliptic forms. The field will be $\mathbb{C}$ at first. Let $H$ be a Hilbert space over $\mathbb{C}$. Whereas before we considered sesquilinear forms which were defined on a second Hilbert space, we now consider sesquilinear forms defined on a domain which carries no Hilbert space structure at the beginning.

Let $D(a) \subset H$ be a subspace of $H$ and let

$$a : D(a) \times D(a) \to \mathbb{C}$$

be sesquilinear. We call $D(a)$ the **domain** of $a$. For $u \in D(a)$ we let

$$a(u) := a(u, u)$$

for short. The **Polarisation Identity**

$$a(u,v) = \frac{1}{4}(a(u+v) - a(u-v) + i(a(u+iv) - a(u-iv))) \tag{8.1}$$

holds for all $u, v \in D(a)$, as one easily verifies. For this it is important that the underlying field is $\mathbb{C}$. In the real case there is no such formula expressing $a(u,v)$ by the diagonal terms $a(u)$.

**Definition 8.2.1.** *The form $a$ is called*

- **densely defined** *if $D(a)$ is dense in $H$,*
- **accretive** *if $\operatorname{Re} a(u) \geq 0$ $(u \in D(a))$,*
- **bounded below** *if $\operatorname{Re} a(u) + \omega\|u\|_H^2 \geq 0$ for all $u \in D(a)$ and some $\omega \in \mathbb{R}$,*
- **symmetric** *if $a(u,v) = \overline{a(v,u)}$ $(u, v \in D(a))$,*
- **positive** *if $a(u) \in \mathbb{R}_+$ $(u \in D(a))$.*

It follows from the polarization identity that the form $a$ is symmetric if and only if $a(u) \in \mathbb{R}$ for all $u \in D(a)$. In particular, each positive form is symmetric.

Assume that $a$ is densely defined. As in Lecture 7 we associate an operator $A$ on $H$ with $a$ by letting

$$D(A) := \{u \in D(a) : \exists f \in H \text{ s.t. } a(u,v) = (f \mid v)_H \text{ for all } v \in D(a)\} \tag{8.2}$$

$$Au := f. \tag{8.3}$$

Notice that for $\omega \in \mathbb{R}$ the operator $A + \omega$ is associated with the form $a + \omega$ given by

$$\begin{aligned} D(a+\omega) &:= D(a), \\ (a+\omega)(u,v) &:= a(u,v) + \omega(u \mid v)_H. \end{aligned}$$

This procedure is just a **rescaling** of the form $a$.

We want to establish conditions on $a$ which imply that $A$ generates a (holomorphic) $C_0$-semigroup on $H$. For this we may consider $a + \omega$ instead of $a$ if necessary, since $A$ generates a (holomorphic) semigroup if and only if $A + \omega$ does so.

Here is the plan for the remainder of this section. We now consider several assumptions on $a$. First we consider the case where $a$ is positive, then the case where $a$ is accretive. Since $a$ is bounded below if and only if $a + \omega$ is accretive for some $\omega \in \mathbb{R}$, we will then obtain also results for $a$ being merely bounded below by rescaling. Up to this moment the field will be $\mathbb{K} = \mathbb{C}$. We finally will interprete the results for bilinear forms considering their sesquilinear extension.

If $a$ is a positive form then

$$(u \mid v)_a := a(u,v) + (u \mid v)_H \qquad (u, v \in D(a))$$

defines a scalar product with corresponding norm

$$\|u\|_a = (a(u) + \|u\|^2)^{\frac{1}{2}} \qquad (u \in D(a)).$$

We say that $a$ is **closed** if the linear space $D(a)$ is complete for the norm $\|\cdot\|_a$. If in addition $D(a)$ is dense in $H$, then we may let $V = D(a)$. Then $V \overset{d}{\hookrightarrow} H$, the form $a : V \times V \to \mathbb{C}$ is continuous, and we may consider the associated operator $A$ on $H$ which is self-adjoint.

We want to extend this to non-symmetric forms. For this the following version of Schwarz's Inequality plays a crucial role.

**Proposition 8.2.2** (Schwarz's Inequality)**.** *Let $a, b$ be sesquilinear forms with the same domain $D(a) = D(b)$. Assume that $b$ is symmetric and that there exists $M \geq 0$ such that*

$$|a(u)| \leq Mb(u) \qquad (u \in D(a)). \tag{8.4}$$

*Then*

$$|a(u,v)| \leq Mb(u)^{\frac{1}{2}} b(v)^{\frac{1}{2}} \qquad (u, v \in D(a)). \tag{8.5}$$

Note that each sesquilinear form $a$ satisfies the **Parallelogram Identity**

$$a(u+v) + a(u-v) = 2a(u) + 2a(v) \qquad (u, v \in D(a)) \tag{8.6}$$

as one easily verifies.

*Proof of Proposition 8.2.2.* Let $u, v \in D(a)$. In order to show (8.5) we may assume that $a(u,v) \in \mathbb{R}$, otherwise replacing $u$ by $e^{-i\theta}u$ for $\theta$ suitable. Then by the Polarisation Identity (8.1)

$$a(u,v) = \frac{1}{4}(a(u+v) - a(u-v)).$$

Hence by assumption

$$|a(u,v)| \leq \frac{M}{4}(b(u+v) + b(u-v)) = \frac{M}{2}(b(u) + b(v)),$$

by virtue of the Parallelogram Identity (8.6).

Let now $\alpha > 0$ and replace $u$ by $\alpha u$ and $b$ by $\frac{1}{\alpha}b$. Then we obtain

$$|a(u,v)| \leq \frac{M}{2}(\alpha^2 b(u) + \frac{1}{\alpha^2} b(v)).$$

If $b(u) \neq 0$ we let $\alpha^2 = (\frac{b(v)}{b(u)})^{\frac{1}{2}}$ and obtain (8.5). If $b(u) = 0$ we let $\alpha^2 \to \infty$ and we see that $a(u, v) = 0$ and (8.5) holds also in this case. $\square$

We continue to consider the sesquilinear form $a : D(a) \times D(a) \to \mathbb{C}$. Define the symmetric forms $a_1, a_2$ by

$$\begin{aligned} a_1 &:= \frac{1}{2}(a + a^*), \qquad D(a_1) := D(a), \\ a_2 &:= \frac{1}{2i}(a - a^*), \qquad D(a_2) := D(a). \end{aligned}$$

Then

$$a = a_1 + ia_2$$

and in particular

$$a_1(u) = \operatorname{Re} a(u) \qquad (u \in D(a)). \tag{8.7}$$

We call $a_1$ the **real** and $a_2$ the **imaginary part** of $a$.

Now we assume that $a$ is accretive, i.e., $\operatorname{Re} a(u) \geq 0$ $(u \in D(a))$. Then

$$(u \mid v)_a := a_1(u, v) + (u \mid v)_H \qquad (u, v \in D(a)) \tag{8.8}$$

defines a scalar product on $D(a)$ with associated norm

$$\|u\|_a = (\operatorname{Re} a(u) + \|u\|_H^2)^{\frac{1}{2}}.$$

**Definition 8.2.3.** *We say that $a$ is* **continuous** *if there exists a constant $c \geq 0$ such that*

$$|\operatorname{Im} a(u)| \leq c\|u\|_a^2 \qquad (u \in D(a)). \tag{8.9}$$

*We say that $a$ is* **closed** *if $a$ is continuous and the space $D(a)$ is complete with respect to the norm $\|\cdot\|_a$.*

Assume that $a$ is closed and densely defined. Then

$$|a(u)| \leq (1 + c)\|u\|_a^2 \qquad (u \in D(a)).$$

Hence by Schwarz's Inequality

$$|a(u, v)| \leq (1 + c)\|u\|_a\|v\|_a \qquad (u, v \in D(a)).$$

Thus, if we consider the Hilbert space $V = D(a)$ endowed with the scalar product $(\cdot \mid \cdot)_a$, the form $a : V \times V \to \mathbb{C}$ is continuous and elliptic. Consequently, the operator $-A$

generates a holomorphic $C_0$-semigroup on $H$ by the results of Section 7.2, where $A$ is the operator associated with $a$.

If $a$ is not accretive, we say that $a$ is **closed** if $a + \omega$ is accretive and closed for some $\omega \in \mathbb{R}$. Thus, by rescaling, we obtain again that $-A$ generates a holomorphic $C_0$-semigroup on $H$.

Finally we consider the case where $\mathbb{K} = \mathbb{R}$. Let $H$ be a real Hilbert space. Let $a : D(a) \times D(a) \to \mathbb{R}$ be a bilinear form whose domain $D(a)$ is a subspace of $H$. If $a$ is accretive (i.e., $a(u) \geq 0$ for all $u \in D(a)$) then we consider the norm

$$\|u\|_a := (a(u) + \|u\|_H^2)^{\frac{1}{2}}.$$

In order to see that $\| \cdot \|_a$ is actually a norm observe that $a(u) = a_1(u)$ where $a_1$ is the symmetric form

$$a_1(u, v) = \frac{1}{2}(a(u, v) + a(v, u)) \qquad (u, v \in D(a)).$$

We say that $a$ is **continuous** if

$$|a(u, v)| \leq M\|u\|_a\|v\|_a \qquad (u, v \in D(a))$$

for some $M > 0$. Finally, the form $a$ is closed if $a$ is continuous and $D(a)$ is complete for the norm $\| \cdot \|_a$.

If $a$ is not necessarily accretive, then we say that $a$ is **closed** if $a + \omega$ is accretive and closed for some $\omega \in \mathbb{R}$.

**Theorem 8.2.4.** *Let $a$ be a closed densely defined form on a real Hilbert space and $A$ the operator associated with $a$. Then $-A$ generates a holomorphic $C_0$-semigroup $(e^{-tA})_{t\geq 0}$ on $H$. The semigroup is contractive if and only if $a$ is accretive.*

*Proof.* The sesquilinear extension $a_{\mathbb{C}}$ of $a$ given by

$$a_{\mathbb{C}}(u_1 + iv_1, u_2 + iv_2) = a(u_1, u_2) + a(v_1, v_2) + i(a(v_1, u_2) - a(u_1, v_2))$$

for all $(u_1 + iu_2), (v_1 + iv_2) \in D(a_{\mathbb{C}}) = D(a) + iD(a)$ is closed. Denote by $(T(t))_{t\geq 0}$ the holomorphic $C_0$-semigroup generated by $-A_{\mathbb{C}}$ on $H_{\mathbb{C}}$, where $A_{\mathbb{C}}$ is the operator associated with $a_{\mathbb{C}}$. Since $(I + tA_{\mathbb{C}})^{-1}H \subset H$ and $(I + tA)^{-1} = (I + tA_{\mathbb{C}})^{-1}_{|H}$ the semigroup $(T(t))_{t\geq 0}$ leaves $H$ invariant. Thus, its part in $H$ is again a holomorphic $C_0$-semigroup and $-A$ is its generator. $\square$

## 8.3 Form sums and multiplicative perturbations

Let $H$ be a Hilbert space over $\mathbb{K}$. Now we have two different but equivalent concepts. The first consists in considering an elliptic, continuous, densely defined form $(a, V)$ on $H$. By this we understand that $V$ is a Hilbert space, $V \overset{d}{\hookrightarrow} H$ and $a : V \times V \to \mathbb{K}$ is continuous and $H$-elliptic. This setting is most convenient for many examples and we will use it frequently. The other is to consider a closed densely defined form $a$ on $H$.

In any case the associated operator $A$ is the negative generator of a holomorphic $C_0$-semigroup on $H$. We illustrate the power of form methods by two perturbation results.

**Theorem 8.3.1.** *Let $(a_1, V_1)$ and $(a_2, V_2)$ be two continuous, elliptic forms on $H$. Consider the form $(a, V)$ on $H$ given by*

$$a(u, v) = a_1(u, v) + a_2(u, v)$$

*defined on $V = V_1 \cap V_2$ with the scalar product*

$$(u \mid v)_V := (u \mid v)_{V_1} + (u \mid v)_{V_2}.$$

*Then $a$ is a continuous, elliptic form. Hence, if $\overline{V} = H$, then the operator $A$ associated with $a$ is the negative generator of a holomorphic $C_0$-semigroup $T$ on $H$.*

Denote by $A_j$ the operator associated with $a_j$, $j = 1, 2$. One calls $A$ the **form sum** of $A_1$ and $A_2$.

*Proof.* One has $\|u\|_V^2 = \|u\|_{V_1}^2 + \|u\|_{V_2}^2$. Hence $V$ is complete. There exist $\omega_1, \omega_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > 0$ such that

$$\begin{aligned} \operatorname{Re} a_1(u) + \omega_1\|u\|_H^2 &\geq \alpha_1\|u\|_{V_1}^2 \qquad (u \in V_1), \\ \operatorname{Re} a_2(u) + \omega_2\|u\|_H^2 &\geq \alpha_2\|u\|_{V_2}^2 \qquad (u \in V_2). \end{aligned}$$

Let $\omega = \omega_1 + \omega_2$ and $\alpha = \min\{\alpha_1, \alpha_2\}$. Then

$$\operatorname{Re} a(u) + \omega\|u\|_H^2 \geq \alpha\|u\|_H^2 \qquad (u \in V).$$

Then $a$ is elliptic. There exists $M > 0$ such that

$$\begin{aligned} |a_1(u, v)| &\leq M\|u\|_{V_1}\|v\|_{V_1} \qquad (u, v \in V_1), \\ |a_2(u, v)| &\leq M\|u\|_{V_2}\|v\|_{V_2} \qquad (u, v \in V_2). \end{aligned}$$

Hence

$$|a(u, v)| \leq M\|u\|_V\|v\|_V \qquad (u, v \in V),$$

thus $a$ is continuous. □

**Example 8.3.2** (unbounded potential)**.** *Let $\mathbb{K} = \mathbb{R}$. Let $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ open. Let*

$$a_1(u, v) = \int_\Omega \nabla u \nabla v dx, \qquad u, v \in V_1 := H_0^1(\Omega),$$

*so that the associated operator is $\Delta_\Omega^D$. Let $c \in L^1_{\mathrm{loc}}(\Omega)$, $c \geq 0$, and*

$$a_2(u,v) = \int_\Omega cuv dx, \qquad u,v \in V_2 := L^2(\Omega, (1+c(x)dx)).$$

*Then, the semigroup associated with $a_2$ is given by $T_2(t)f = e^{-tc}f$.*

*Let $a = a_1 + a_2$ on $V = V_1 \cap V_2$. Then $\overline{V} = L^2(\Omega)$ and the operator associated with $a$ is given by*

$$\begin{aligned} D(A) &= \{u \in H_0^1(\Omega) \cap V_2 : \ -\Delta u + cu \in L^2(\Omega)\}, \\ Au &= \Delta u + cu, \end{aligned}$$

*(where, for $u \in L^2(\Omega)$, $-\Delta u + cu \in L^2(\Omega)$ is defined as an element of $\mathcal{D}(\Omega)'$).*

If $(a, V)$ is a densely defined, continuous, elliptic form on $H$, then the definition of the operator $A$ on $H$ associated with $a$ depends crucially on the scalar product considered on $H$. If we consider another equivalent scalar product, then we obtain a different operator. This can be used to prove the following perturbation result.

**Theorem 8.3.3** (multiplicative perturbation)**.** *Let $(a, V)$ be a densely defined, continuous, elliptic form on $H$. Denote by $A$ the associated operator on $H$. Let $S \in \mathcal{L}(H)$ be selfadjoint such that*

$$(Sx \mid x)_H \geq \delta \|x\|_H^2 \qquad (x \in H),$$

*where $\delta > 0$. Then $-SA$ generates a holomorphic $C_0$-semigroup.*

*Proof.* Consider the scalar product

$$(u \mid v)_1 := (S^{-1}u \mid v)_H \qquad (u \in H)$$

on $H$. It induces an equivalent norm on $H$. We let $H_1$ be the space $H$ endowed with this new scalar product. Then $(a, V)$ is also continuous and elliptic on $H_1$. Let $A_1$ be the operator on $H_1$ that is associated with $(a, V)$. We show that

$$A_1 = SA.$$

Let $u \in V$, $f \in H$. Then $u \in D(A_1)$ and $A_1 u = f$ if and only if

$$a(u,v) = (f \mid v)_{H_1} = (S^{-1}f \mid v)_H \qquad (u \in V)$$

if and only if $u \in D(A)$ and $Au = S^{-1}f$. This proves the claim. □

**Example 8.3.4** (multiplicative perturbation of the Dirichlet Laplacian)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $m : \Omega \to \mathbb{R}$ be bounded and measurable such that $m(x) \geq \delta$ for some $\delta > 0$ and a.e. $x \in \Omega$. Then $m\Delta_\Omega^D$ generates a holomorphic semigroup on $L^2(\Omega)$.*

## 8.4 Exercises

At first we consider similarity of forms. The following exercise shows in particular that Example 8.1.4 yields the most general example of a selfadjoint semigroup, up to rescaling and to unitary equivalence.

**Exercise 8.4.1** (similarity of forms). *a) Let $H_1, H_2, V_1$ be Hilbert spaces over $\mathbb{K}$, $V_1 \overset{d}{\hookrightarrow} H_1$. Let $a_1 : V_1 \times V_1 \to \mathbb{K}$ be a continuous $H_1$-elliptic form. Denote by $(T_1(t))_{t\geq 0}$ the $C_0$-semigroup associated with $a_1$. Let $U : H_1 \to H_2$ be unitary. Show that the semigroup $(T_2(t))_{t\geq 0}$ given by*

$$T_2(t) := UT_1(t)U^{-1} \qquad (t \geq 0)$$

*is associated with a densely defined, continuous, elliptic form on $H_2$.*

*b) Let $-A$ be a symmetric m-dissipative operator on a separable Hilbert space $H$. Deduce from a) that $A$ is associated with a densely defined, symmetric, continuous, $H$-elliptic form.*

In the following exercise a common procedure is extended from bounded to unbounded operators.

**Exercise 8.4.2** (the operator $A^*A$). *Let $A$ be a closed, densely defined operator on a Hilbert space $H$. Show that the operator $-A^*A$ defined on*

$$D(A^*A) := \{x \in D(A) : Ax \in D(A^*)\}$$

*is m-dissipative and symmetric.*

**Hint:** consider the form $a(u, v) := (Au \mid Av)$ $(u, v \in D(a) := D(A))$.

Let $a : D(a) \times D(a) \to \mathbb{C}$ be a sesquilinear form on a Hilbert space $H$. The set

$$W(a) := \{a(u) \in \mathbb{C} : u \in D(a),\ \|u\|_H = 1\}$$

is called the **numerical range** of $a$. The following exercise explains why continuous forms are sometimes also called **sectorial** (cf. [Kat66]).

**Exercise 8.4.3** (continuity of forms). *The following are equivalent*

*i) There exists $\omega \geq 0$ such that $a + \omega$ is accretive and continuous.*

*ii) There exist $c \geq 0$ and $\omega \geq 0$ such that*

$$|\operatorname{Im} a(u)| \leq c(\operatorname{Re} a(u) + \omega\|u\|_H^2) \qquad (u \in D(a)).$$

*iii) There exist $c \geq 0$ and $\omega \geq 0$ such that*

$$|\operatorname{Im}(a + \omega)(u)| \leq c \operatorname{Re}(a + \omega)(u) \qquad (u \in D(a)).$$

*iv) There exist $\omega \geq 0$ and $\alpha \in (0, \frac{\pi}{2})$ such that $W(a + \omega) \subset \Sigma_\alpha$, where $\Sigma_\alpha$ denotes the sector of angle $\alpha$ introduced in Section 2.5.*

**Exercise 8.4.4** (continuity of bilinear forms). *Let $H$ be a real Hilbert space and $a : D(a) \times D(a) \to \mathbb{R}$ be bilinear and accretive. Recall the definition of the norm $\|\cdot\|_a$. Then the following are equivalent.*

*i) There exists $c \geq 0$ such that*

$$|a(u, v) - a(v, u)| \leq c\|u\|_a\|v\|_a \qquad (u, v \in D(a)).$$

*ii) There exists $d \geq 0$ such that*

$$|a(u, v)| \leq d\|u\|_a\|v\|_a \qquad (u, v \in D(a)).$$

## 8.5 Comments

A natural question arises: Which are the holomorphic semigroups which are associated with a form? Here is the answer.

### 8.5.1 Characterisation of operators associated with a form

Let $H$ be a complex Hilbert space and $A$ be a closed operator. The following are equivalent.

i) There exists a continuous, elliptic, densely defined form $(a, V)$ such that $A$ is associated with $(a, V)$.

ii) There exists $\omega \in \mathbb{R}$, $\alpha \in (0, \frac{\pi}{2})$ such that $W(A) \subset \omega + \Sigma_\alpha$ and $\rho(A) \setminus (\omega + \Sigma_a) \neq \emptyset$.

Here $W(A)$ denotes the **numerical range**

$$W(A) := \{(Au \mid u)_H : \ u \in D(A), \ \|u\|_H = 1\}$$

of the operator $A$. See [Kat66] for a proof.

This characterisation holds if we consider a fixed scalar product on $H$. If we allow also equivalent scalar product on $H$ (as in Theorem 8.3.3) the characterisation is more difficult (see [Are04] and the references given there).

### 8.5.2 The square root problem

Let $-A$ be $m$-dissipative on a Hilbert space $H$. Then there exists a unique operator $B$ on $H$ such that $-B$ is $m$-dissipative and $B^2 = A$ (see [Kat66]): $B$ is called the **square root** of $A$ and denoted by $A^{\frac{1}{2}}$. Now assume that $A$ is associated with a densely defined, continuous, elliptic form $(a, V)$. If $a$ is symmetric it is easy to see from the Spectral Theorem that $V = D(A^{\frac{1}{2}})$. However, this is no longer true in general. A counterexample is due to McIntosh [McI82]. It was an open problem for long time whether $V = D(A^{\frac{1}{2}})$ for the elliptic operator considered in Example 7.1.9 on $\mathbb{R}^n$. This is known as Kato's problem. It was finally solved by sophisticated tools mainly from Harmonic Analysis by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [AHLMT02].

# Lecture 9

# Invariance of Closed Convex Sets and Positivity

We continue to consider an elliptic, continuous and densely defined form on a Hilbert space $H$ and the associated semigroup. The aim of this lecture is to describe in terms of the form when the semigroup leaves invariant a closed, convex subset of $H$. In particular, we will find conditions for positivity and for the semigroup being submarkovian (the Beurling–Deny criteria).

## 9.1 Invariance of closed, convex sets

Let $(a, V)$ be a continuous, elliptic, densely defined form on a Hilbert space $H$ over $\mathbb{K}$. Let $M \geq 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \qquad (u,v \in V). \tag{9.1}$$

We will later assume that $a$ is accretive, i.e., $\operatorname{Re} a(u) \geq 0$ $(u \in V)$. Then, by the ellipticity of $a$, there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \|u\|_H^2 \geq \alpha\|u\|_V^2 \qquad (u \in V). \tag{9.2}$$

This means that

$$\|u\|_a := (\operatorname{Re} a(u) + \|u\|_H^2)^{\frac{1}{2}} \qquad (u \in V)$$

defines an equivalent norm on $V$, cf. Lecture 8.

We denote by $A$ the operator associated with $a$ and by $(e^{-tA})_{t\geq 0}$ the semigroup generated by $-A$. Let $C$ be a non-empty closed convex subset of $H$. We denote by $P$ the orthogonal projection of $H$ onto $C$. Recall that $P : H \to H$ is characterized as follows.

**Proposition 9.1.1.** *Let $u \in H$, $v_0 \in C$. Then*

$$Pu = v_0 \tag{9.3}$$

*if and only if*

$$\|u - v_0\| = \min_{v \in C} \|u - v\| \tag{9.4}$$

*if and only if*

$$\operatorname{Re}(u - v_0 \mid v - v_0) \le 0 \qquad \text{for all } v \in C. \tag{9.5}$$

Condition (9.4) says that the distance of $u$ to $C$ is minimal at the point $v_0$, whereas condition (9.5) says that the angle between $u - v_0$ and $v - v_0$ is larger than or equal to $\frac{\pi}{2}$ for all $v \in C$.

Invariance under the semigroup and under the resolvent are equivalent. In fact, for this the Hilbert space structure is not needed.

**Proposition 9.1.2.** *Let $-A$ be the generator of a contractive $C_0$-semigroup on a Banach space $X$ and let $C \subset X$ be closed and convex. The following assertions are equivalent.*

*i)* $e^{-tA}C \subset C$ *for all* $t \ge 0$;

*ii)* $\lambda(\lambda + A)^{-1}C \subset C$ *for all* $\lambda > 0$.

*Proof.* *i)* $\Longrightarrow$ *ii)* Assume that there exist $u \in C$, $\lambda > 0$, such that $\lambda(\lambda + A)^{-1}u \notin C$. By the Hahn–Banach Theorem there exists a continuous functional $\phi$ on $X$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}\phi(\lambda(\lambda + A)^{-1}u) > \alpha > \operatorname{Re}\phi(v) \qquad \text{for all } v \in C. \tag{9.6}$$

Observe that

$$\lambda(\lambda + A)^{-1}u = \int_0^\infty \lambda e^{-\lambda t} e^{-At} u \, dt. \tag{9.7}$$

Applying (9.6) to $v = e^{-tA}u$ we obtain

$$\begin{aligned} \operatorname{Re}\phi(\lambda(\lambda + A)^{-1}u) \;&>\; \alpha = \int_0^\infty \lambda e^{-\lambda t} \alpha \, dt \\ &\ge\; \int_0^\infty \lambda e^{-\lambda t} \operatorname{Re}\phi(e^{-tA}u) \, dt \\ &=\; \operatorname{Re}\phi\left(\int_0^\infty \lambda e^{-\lambda t} e^{-tA} u \, dt\right) \\ &=\; \operatorname{Re}\phi(\lambda(\lambda + A)^{-1}u), \end{aligned}$$

which is a contradiction.

*ii)* $\Longrightarrow$ *i)* Since $e^{-tA}u = \lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}u$ for all $u \in C$, the claim follows from *ii*). $\square$

We will use the following simple argument, where we write $u_n \rightharpoonup u$ to indicate that $(u_n)_{n\in\mathbb{N}}$ converges weakly to $u$.

**Lemma 9.1.3.** *Let $u_n \in V$ such that $u_n \rightharpoonup u$ in $H$ and $\sup_{n\in\mathbb{N}} \|u_n\|_V < \infty$. Then $u \in V$.*

*Proof.* Since $V$ is reflexive, there are a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and $v \in V$ such that $u_{n_k} \rightharpoonup v$ in $V$. But then $u_{n_k} \rightharpoonup v$ in $H$ since $V \hookrightarrow H$. Hence $u = v \in V$. $\square$

**Lemma 9.1.4.** *Let $u, v \in V$. Then*

$$a(u,v) = \lim_{t\downarrow 0} \frac{1}{t}\left(u - (I+tA)^{-1}u \mid v\right)_H .$$

*Proof.* Recall from Section 8.1 that the operator $\mathcal{A} : V \to V'$ given by $< \mathcal{A}u, v >= a(u,v)$ generates a $C_0$-semigroup on $V'$. Hence $\lim_{t\downarrow 0}(I + t\mathcal{A})^{-1}f = f$ in $V'$ for all $f \in V'$. Consequently, for all $u, v \in V$

$$\begin{aligned} \frac{1}{t}\left(u - (I+tA)^{-1}u \mid v\right)_H &= (A(I+tA)^{-1}u \mid v)_H \\ &= < (I+t\mathcal{A})^{-1}\mathcal{A}u, v > \to < \mathcal{A}u, v >= a(u,v). \end{aligned}$$

as $t \downarrow 0$. $\square$

The following is the main result in this section.

**Theorem 9.1.5.** *Assume that $a$ is accretive. Let $C$ be a closed, convex subset of $H$ and $P$ the orthogonal projection of $H$ onto $C$. The following are equivalent.*

*i)* $e^{-tA}C \subset C \quad (t \geq 0)$,

*ii)* $PV \subset V$ *and* $\operatorname{Re} a(Pu, u - Pu) \geq 0 \quad (u \in V)$,

*iii)* $PV \subset V$ *and* $\operatorname{Re} a(u, u - Pu) \geq 0 \quad (u \in V)$.

*Proof.* Let $I_t := (I + tA)^{-1}$, $t > 0$.

*i)* $\Longrightarrow$ *ii)*: a) Let $u \in V$. We show that $Pu \in V$.
Since $a$ is accretive, we may assume that $\|v\|_V^2 = \|v\|_a^2 = \operatorname{Re} a(v) + \|v\|_H^2$ for all $v \in V$.
Since $I_t + tAI_t = I$, we have

$$\begin{aligned} \operatorname{Re} a(I_tPu, I_tPu) &= \operatorname{Re}(AI_tPu \mid I_tPu)_H = \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid I_tPu)_H \\ &= \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid I_tPu - Pu)_H + \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid Pu)_H \\ &\leq \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid Pu)_H \\ &= \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid Pu - u)_H + \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid u)_H \\ &\leq \frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid u)_H \end{aligned}$$

by (9.5) since $I_tPu \in C$ by assumption and Proposition 9.1.2. Thus,

$$\begin{aligned}
\operatorname{Re} a(I_tPu, I_tPu) &\leq \operatorname{Re}(AI_tPu \mid u)_H = \operatorname{Re} a(I_tPu, u) \leq M\|I_tPu\|_V\|u\|_V \\
&\leq \frac{1}{2}\|I_tPu\|_V^2 + \frac{M^2}{2}\|u\|_V^2 \\
&= \frac{1}{2}\operatorname{Re} a(I_tPu, I_tPu) + \frac{1}{2}\|I_tPu\|_H^2 + \frac{M^2}{2}\|u\|_V^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{1}{2}\operatorname{Re} a(I_tPu, I_tPu) &\leq \frac{1}{2}\|I_tPu\|_H^2 + \frac{1}{2}M^2\|u\|_V^2 \\
&\leq \frac{1}{2}\|Pu\|_H^2 + \frac{1}{2}M^2\|u\|_V^2.
\end{aligned}$$

Hence $\sup_{0<t\leq 1}\|I_tPu\|_V < \infty$. Since $I_tPu \to Pu$ in $H$ as $t \downarrow 0$ we get from Lemma 9.1.3 that $Pu \in V$.

b) Let $u \in V$. Then $Pu \in V$ by a). By Lemma 9.1.4 we get

$$\operatorname{Re} a(Pu, u - Pu) = \lim_{t\to 0}\frac{1}{t}\operatorname{Re}(Pu - I_tPu \mid u - Pu)_H.$$

It follows from (9.5) that $\operatorname{Re} a(Pu, u - Pu) \geq 0$.

*ii)* $\Longrightarrow$ *iii)* Let $u \in V$. Then by the accretivity of $a$

$$\begin{aligned}
\operatorname{Re} a(u, u - Pu) &= \operatorname{Re} a(u - Pu, u - Pu) + \operatorname{Re} a(Pu, u - Pu) \\
&\geq \operatorname{Re} a(Pu, u - Pu) \geq 0.
\end{aligned}$$

*iii)* $\Longrightarrow$ *i)* Let $u \in C$, $t > 0$. Then

$$\begin{aligned}
\|I_tu - PI_tu\|_H^2 &= (I_tu - PI_tu \mid I_tu - PI_tu)_H \\
&= \operatorname{Re}(I_tu - u \mid I_tu - PI_tu)_H + \operatorname{Re}(u - PI_tu \mid I_tu - PI_tu)_H \\
&= -t\operatorname{Re}(AI_tu \mid I_tu - PI_tu)_H + \operatorname{Re}(u - PI_tu \mid I_tu - PI_tu)_H \\
&\leq \operatorname{Re}(u - PI_tu \mid I_tu - PI_tu)_H
\end{aligned}$$

by assumption *iii)*. Thus,

$$\|I_tu - PI_tu\|_H^2 \leq 0,$$

since $u \in C$. Hence $I_tu = PI_tu \in C$ for all $t > 0$. Proposition 9.1.2 implies *i)*. □

## 9.2 Positivity

Let $H = L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. We let $\mathbb{K} = \mathbb{R}$ in this section. Let $(a, V)$ be a continuous, densely defined elliptic form on $H$ and $A$ the associated operator on $H$.

**Theorem 9.2.1** (First Beurling–Deny Condition)**.** *The following are equivalent.*

*i)* $e^{-tA} \geq 0 \quad (t \geq 0)$,

*ii)* $u \in V$ *implies* $u^+ \in V$ *and* $a(u^+, u^-) \leq 0$.

*Proof.* The orthogonal projection $P$ of $H$ onto $H_+ := \{f \in L^2(\Omega) : f \geq 0 \text{ a.e. }\}$ is given by $Pu := u^+$ as is easy to see.

Let now $\omega \in \mathbb{R}$ and observe that on one hand

$$e^{-tA} \geq 0 \qquad \text{if and only if} \qquad e^{-t(A+\omega)} = e^{-tA}e^{-\omega t} \geq 0$$

and on the other

$$(a + \omega)(u^+, u^-) = a(u^+, u^-) + \omega(u^+, u^-)_H = a(u^+, u^-).$$

Thus we may assume that $a$ is accretive, replacing $a$ by $a + \omega$ otherwise. Since $a(Pu, u - Pu) = -a(u^+, u^-)$, the claim follows from Theorem 9.1.5. $\square$

**Example 9.2.2.** *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $V := H_0^1(\Omega)$,

$$a(u, v) := \int_\Omega \left( \sum_{i,j=1}^n a_{i,j} D_i u D_j v + \sum_{j=1}^n b_j D_j u v + cuv \right) dx$$

*where* $a_{ij}, b_j, c \in L^\infty(\Omega)$, $1 \leq i, j \leq n$ *and*

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for a.e.* $x \in \Omega$ *and some* $\alpha > 0$.

*We have seen in Lecture 7 that* $a$ *is a continuous, densely defined elliptic form. Denote by* $A$ *the associated operator. By Proposition 3.2.1* $u \in H_0^1(\Omega)$ *implies* $u^+, u^- \in H_0^1(\Omega)$, *and* $D_j u^+ = 1_{\{u>0\}} D_j u$, $D_j u^- = -1_{\{u<0\}} D_j u$. *Hence* $a(u^+, u^-) = 0$. *It follows from Theorem 9.2.1 that* $e^{-tA} \geq 0$ *for all* $t \geq 0$.

## 9.3 Submarkovian semigroups

Let $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, $\mathbb{K} = \mathbb{R}$. Let $(a, V)$ be a densely defined, continuous, elliptic form on $H$ with associated operator $A$ on $H$. Recall that the semigroup $(e^{-tA})_{t\geq 0}$ is called **submarkovian** if

$$f \leq 1 \text{ a.e.} \qquad \text{implies} \qquad e^{-tA}f \leq 1 \text{ a.e.} \quad (t \geq 0)$$

for all $f \in H$. This implies in particular that $e^{-tA} \geq 0$ for all $t \geq 0$.

Let $C := \{u \in L^2(\Omega) : u \leq 1\}$. Then $C$ is clearly convex and closed. Note that for $u \in L^2(\Omega)$

$$u = u \wedge 1 + (u-1)^+, \tag{9.8}$$

$$1 = u \wedge 1 + (1-u)^+. \tag{9.9}$$

**Lemma 9.3.1.** *The orthogonal projection $P$ onto $C$ is given by $Pu = u \wedge 1$.*

*Proof.* Let $u \in H$. We have to show that $(u - u \wedge 1 \mid v - u \wedge 1)_H \leq 0$ for all $v \in C$. By (9.8) and (9.9) we see that

$$\begin{aligned} (u - u \wedge 1 \mid v - u \wedge 1)_H &\leq (u - u \wedge 1 \mid 1 - u \wedge 1)_H \\ &= \big((u-1)^+ \mid (1-u)^+\big)_H = 0. \end{aligned}$$

This concludes the proof. ☐

**Theorem 9.3.2** (Second Beurling–Deny Condition)**.** *Assume that $a$ is accretive. Then the following are equivalent.*

*i)* $(e^{-tA})_{t\geq 0}$ *is submarkovian,*

*ii)* $u \in V$ *implies* $u \wedge 1 \in V$ *and* $a(u \wedge 1, (u-1)^+) \geq 0$.

Note that for $u \in V$, $u \wedge 1 \in V$ implies $(u-1)^+ \in V$ since $u \wedge 1 + (u-1)^+ = u$. Let $Pu = u \wedge 1$. Then $a(Pu, u - Pu) = a(u \wedge 1, (u-1)^+)$. So Theorem 9.3.2 follows from Theorem 9.1.5.

Let $B \in \mathcal{L}(L^2(\Omega))$. Then

$$B \geq 0 \qquad \text{and} \qquad \|Bf\|_1 \leq \|f\|_1 \quad (f \in L^1 \cap L^2) \tag{9.10}$$

if and only if $B^*$ is submarkovian. Thus, we obtain the following corollary.

**Corollary 9.3.3.** *Assume that $a$ is accretive. The following are equivalent.*

*i)* $(e^{-tA})_{t\geq 0}$ *is positive and* $\|e^{-tA}f\|_1 \leq \|f\|_1$ $(t \geq 0,\ f \in L^1 \cap L^2)$,

*ii)* $u \in V$ *implies* $u \wedge 1 \in V$ *and* $a((u-1)^+, u \wedge 1) \geq 0$.

**Example 9.3.4.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $a_{ij}, b_j, c \in L^\infty(\Omega)$, $1 \leq i, j \leq n$, *be such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for a.e.* $x \in \Omega$ *and some* $\alpha > 0$. *Let* $V := H_0^1(\Omega)$ *and*

$$a(u,v) := \int_\Omega \left( \sum_{i,j=1}^{n} a_{i,j} D_i u D_j v + \sum_{j=1}^{n} b_j D_j u v + cuv \right) dx.$$

*We have shown that* $a$ *is continuous and elliptic. Let* $A$ *be the operator associated with* $A$. *We already know that* $e^{-tA} \geq 0$ $(t \geq 0)$. *The following further assertions hold.*

*a) The semigroup* $(e^{-tA})_{t\geq 0}$ *is submarkovian if* $c \geq 0$;

*b) Assume that* $b_j \in C^1(\Omega)$, $D_j b_j \in L^\infty(\Omega)$, *and* $\sum_{j=1}^n D_j b_j \leq c$, *then*

$$\|e^{-tA}f\|_1 \leq \|f\|_1 \qquad (t > 0,\ f \in L^1 \cap L^2);$$

*c) As a consequence, if* $c \geq 0$ *and* $\sum_{j=1}^n D_j b_j \leq c$, *then there exists a positive consistent semigroup* $(T_p(t))_{t\geq 0}$ *on* $L^p(\Omega)$, $1 \leq p \leq \infty$, *such that* $(T_p(t))_{t\geq 0}$ *is a* $C_0$*-semigroup for* $p < \infty$, $T_\infty$ *is a dual semigroup,* $T_2(t) = e^{-tA}$ $(t \geq 0)$, *and* $\|T_p(t)\|_{\mathcal{L}(L^p(\Omega))} \leq 1$ $(t \geq 0)$.

*Proof.* a) Let $u \in H_0^1(\Omega)$. Then $D_j(u \wedge 1) = D_j u\, 1_{u\leq 1}$, $D_j(u-1)^+ = D_j u\, 1_{u \geq 1}$. Observe that by Stampacchia's Lemma (Corollary 3.2.2) we have

$$D_j u(x) = 0 \qquad \text{a.e. on } \{u = 1\}. \tag{9.11}$$

Thus

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij} D_i(u \wedge 1) D_j(u-1)^+ = 0$$

and

$$\int_\Omega \sum_{j=1}^{n} b_j D_j(u \wedge 1)(u-1)^+ = 0.$$

If $c \geq 0$, it follows that $a(u \wedge 1, (u-1)^+) \geq 0$. Thus, a) follows now from Theorem 9.3.2.

b) Let $u \in H_0^1(\Omega)$. By Corollary 9.3.3 it suffices to show that $a((u-1)^+, u \wedge 1) \geq 0$. By a) and the definition of the derivative in $H_0^1(\Omega)$ we have that for $u \in H_0^1(\Omega)$

$$\begin{aligned} a((u-1)^+, u \wedge 1) &= \int_\Omega \sum_{j=1}^n b_j D_j (u-1)^+ (u \wedge 1) + c(u-1)^+ (u \wedge 1)\, dx \\ &= \int_\Omega \sum_{j=1}^n D_j \big(b_j (u-1)^+\big)(u \wedge 1)\, dx \\ &\qquad + \int_\Omega \left( -\sum_{j=1}^n D_j b_j (u-1)^+ (u \wedge 1) + c(u-1)^+ (u \wedge 1) \right) dx \\ &\geq \int_\Omega \sum_{j=1}^n D_j \big(b_j (u-1)^+\big)(u \wedge 1) dx \end{aligned}$$

by assumption. Thus,

$$a((u-1)^+, u \wedge 1) \geq -\int_\Omega \sum_{j=1}^n b_j (u-1)^+ D_j (u \wedge 1) dx = 0$$

by (9.11). □

## 9.4 Exercises

We did treat the cases of real and complex Hilbert spaces simultaneously. Alternatively, one can work on complex Hilbert spaces throughout and use the following exercise to obtain results on real Hilbert spaces.

**Exercise 9.4.1** (invariance of the real space)**.** *Let $H$ be a real Hilbert space and let $H_{\mathbb{C}}$ its complexification. Let $(a, V)$ be an elliptic, continuous, densely defined form on $H_{\mathbb{C}}$ with associated operator $A$ on $H_{\mathbb{C}}$. Characterise in terms of $(a, V)$ that*

$$e^{-tA} H \subset H \qquad (t \geq 0).$$

Next we consider perturbation by a potential (i.e., an unbounded multiplication operator). We let $\mathbb{K} = \mathbb{R}$ in the following two exercises.

**Exercise 9.4.2** (additive perturbation by a potential)**.** *Let $(a, V)$ be a continuous, densely defined, elliptic form on $L^2(\Omega)$ which satisfies the first Beurling–Deny condition. Let $c : \Omega \to [0, \infty]$ be measurable such that $V_1 := V \cap \{u \in L^2(\Omega) : \int_\Omega u^2 c < \infty\}$ is dense in $L^2(\Omega)$. Observe that $V_1$ is a Hilbert space with respect to the scalar product*

$$(u, v)_{V_1} := (u, v)_V + (u, v)_{L^2(\Omega, cdx)}.$$

*a) Show that the form $b : V_1 \times V_1 \to \mathbb{R}$ given by*

$$b(u,v) := a(u,v) + \int_\Omega cuv\,dx \qquad (u,v \in V_1)$$

*satisfies the first Beurling–Deny criterion.*

*b) Assume that $a$ is accretive and that the semigroup associated with $a$ is submarkovian. Show that also the semigroup associated with $b$ is submarkovian.*

**Exercise 9.4.3** (multiplicative perturbation)**.** *Let $(a, V)$ be a densely defined, continuous, elliptic form on $L^2(\Omega)$ with associated operator $A$. Let $m \in L^\infty(\Omega)$ such that $m(x) \geq \delta > 0$ a.e.*

*a) Assume that $e^{-tA} \geq 0$ $(t \geq 0)$. Show that $e^{-t(mA)} \geq 0$ $(t \geq 0)$.*
*Hint: Theorem 8.3.3*

*b) Assume that $\|e^{-tA}\|_{\mathcal{L}(L^2(\Omega))} \leq 1$ and that $(e^{-tA})_{t\geq 0}$ is submarkovian. Show that $(e^{-t(mA)})_{t\geq 0}$ is submarkovian.*

**Exercise 9.4.4** (invariance criterion for symmetric forms)**.** *Let $H$ be a Hilbert space over $\mathbb{K} = \mathbb{R}$ and $(a, V)$ a densely defined, continuous, elliptic, accretive, symmetric form with associated operator $A$. Let $P$ be the orthogonal projection onto a closed, convex subset $C$ of $H$. Show that $e^{-tA}C \subset C$ $(t \geq 0)$ if and only if*

$$u \in V \qquad \text{implies} \qquad Pu \in V \text{ and } a(Pu) \leq a(u). \tag{9.12}$$

*Hint: Use the Cauchy–Schwarz inequality*

$$|a(u,v)| \leq a(u)^{\frac{1}{2}} a(v)^{\frac{1}{2}}.$$

## 9.5 Comments

The invariance criterion Theorem 9.1.5 is due to Ouhabaz [Ouh96]. The Beurling–Deny criteria are classical in the symmetric case (see [FOT94], [BH91], [Dav89]); for non-symmetric forms they are due to Ouhabaz [Ouh92a] and [Ouh92b]. Related results are also contained in the book of Ma and Röckner [MaRö92]. Our formulation of the second Beurling–Deny criterion, Theorem 9.3.2, is different than in [Ouh05], since we do not assume positivity.

# Lecture 10

# Irreducible Semigroups and Perron–Frobenius Theory

*Dedicated to the memory of*
*H.H. Schaefer*
*14.02.1925-16.12.2005*

In this lecture we introduce the notion of irreducibility for a positive semigroup. Physically it signifies that heat conduction reaches each point instantaneously. A very simple criterion allows us to establish irreducibility if the semigroup is associated with a continuous elliptic form. Our main attention is given to the case where the resolvent is compact. Then we prove the existence of a unique positive eigenfunction. Moreover, we prove a typical result of Perron–Frobenius Theory, namely that the peripheral boundary spectrum is cyclic. If the semigroup is analytic, then we obtain a dominant eigenvalue. As a consequence we can show that the semigroup converges to an equilibrium. This is our final goal in this lecture. Later we will see how all these results can be applied to elliptic operators. There are four sections in this lecture.

- Irreducible semigroups
- Positive eigenfunctions
- Dominant eigenvalues
- Asymptotic behaviour.

## 10.1 Irreducible semigroups

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $1 \leq p < \infty$. Let $X$ be the real or complex space $L^p(\Omega)$. For $\omega \in \Sigma$ we consider the space

$$L^p(\omega) := \{f \in L^p(\Omega) : f = 0 \text{ a.e. on } \Omega \setminus \omega\}.$$

**Definition 10.1.1.** *A $C_0$-semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega)$ is called* **irreducible** *if, for each $\omega \in \Sigma$, $T(t)L^p(\omega) \subset L^p(\omega)$ $(t > 0)$ implies that $\mu(\omega) = 0$ or $\mu(\Omega \setminus \omega) = 0$.*

For a measurable function $f : \Omega \to \mathbb{R}$ we define

$$\begin{aligned} f \geq 0 \quad &:\Longleftrightarrow \quad f(x) \geq 0 \text{ a.e.}, \\ f > 0 \quad &:\Longleftrightarrow \quad f(x) \geq 0 \text{ a.e. and } \mu(\{x \in \Omega : f(x) \neq 0\}) > 0, \\ f \gg 0 \quad &:\Longleftrightarrow \quad f(x) > 0 \text{ a.e.}. \Longleftrightarrow: f \text{ is strictly positive.} \end{aligned}$$

**Theorem 10.1.2.** *Let $(T(t))_{t\geq 0}$ be a positive, irreducible $C_0$-semigroup on $L^p(\Omega)$, where $1 \leq p < \infty$. If $(T(t))_{t\geq 0}$ is holomorphic, then for all $0 < f \in L^p(\Omega)$*

$$T(t)f \gg 0 \qquad (t > 0).$$

Note that irreducibility merely means that for each $f > 0$, which is not strictly positive, there exists $t > 0$ such that $\mu(\{x \in \Omega \setminus \omega : (T(t)f)(x) \neq 0\}) > 0$, where $\omega := \{x \in \Omega : f(x) > 0\}$. The theorem says that for holomorphic semigroups this already implies that $T(t)f$ is strictly positive for all $t > 0$. If $(T(t))_{t\geq 0}$ is generated by an elliptic operator, then $T(\cdot)f$ is the solution of the Cauchy problem for the corresponding heat equation. Thus the result signifies that heat conduction arrives instantaneously at each point of $\Omega$ even if the initial heat is concentrated on a small region.

**Theorem 10.1.3** (Uniqueness Theorem)**.** *Let $X$ be a Banach space, $D \subset \mathbb{C}$ be an open connected set, and let $f : D \to X$ be holomorphic. Let $Y \subset X$ be a closed subspace. Assume there exists $z_0 \in D$, $z_n \neq z_0$, $\lim_{n\to\infty} z_n = z_0$ such that $f(z_n) \in Y$ for all $n \in \mathbb{N}$. Then $f(z) \in Y$ for all $z \in D$.*

*Proof.* Assume that for some $w \in D$ one has $f(w) \notin Y$. By the Hahn–Banach Theorem there exists $x' \in X'$ such that $x'_{|Y} = 0$ and $\langle f(w), x' \rangle \neq 0$. Then $\langle f(z_n), x' \rangle = 0$ for all $n \in \mathbb{N}$. Since $\langle f(\cdot), x' \rangle$ is a holomorphic function, this contradicts the classical Uniqueness Theorem for scalar-valued holomorphic functions. $\square$

Theorem 10.1.2 is an immediate consequence of the following lemma.

**Lemma 10.1.4.** *Let $(T(t))_{t\geq 0}$ be a positive, holomorphic $C_0$-semigroup on $L^p(\Omega)$ where $1 \leq p < \infty$. Let $\omega \in \Sigma$, $0 < f \in L^p(\Omega)$, $t_0 > 0$. If $T(t_0)f \in L^p(\omega)$, then $T(t)f \in L^p(\omega)$ for all $t > 0$.*

*Proof.* Let $t_n \downarrow 0$ such that $\|T(t_n)f - f\| \le 2^{-n}$. Let $h_n := f - \sum_{k=n}^{\infty}(f - T(t_k)f)^+$. Then $h_n \to f$ as $n \to \infty$. Let $n \in \mathbb{N}$. Then for $m \ge n$ one has

$$h_n \le f - (f - T(t_m)f)^+ = f \wedge T(t_m)f.$$

Hence

$$0 \le T(t_0 - t_m)h_n^+ \le T(t_0 - t_m)T(t_m)f = T(t_0)f \qquad (m \ge n).$$

Thus $T(t_0 - t_m)h_n^+ \in L^p(\omega)$ for all $m \ge n$. It follows from the Uniqueness Theorem 10.1.3 that $T(t)h_n^+ \in L^p(\omega)$ for all $t > 0$. Since $h_n^+ \to f$ as $n \to \infty$, it follows that $T(t)f \in L^p(\omega)$ for all $t > 0$. □

Now we assume that $p = 2$ and let $H = L^2(\Omega)$. Let $(a, V)$ be a densely defined, continuous and elliptic form on $H$ with associated operator $A$. Assume that $u \in V$ implies $u^+, u^- \in V$ and $a(u^+, u^-) \le 0$, so that the semigroup $(e^{-tA})_{t\ge 0}$ is positive by the first Beurling–Deny criterion.

**Theorem 10.1.5.** *Assume that for each $\omega \in \Sigma$*

$$1_\omega V \subset V \text{ implies } \mu(\omega) = 0 \text{ or } \mu(\Omega \setminus \omega) = 0.$$

*Then the semigroup $(e^{-tA})_{t\ge 0}$ is irreducible.*

*Proof.* Replacing $a$ by $a + \lambda$ for suitable $\lambda \ge 0$ we may assume that $a$ is accretive. Let $\omega \in \Sigma$. Then the orthogonal projection $P$ onto $L^2(\omega)$ is given by $Pf = 1_\omega f$ ($f \in L^2(\Omega)$). Assume that the semigroup leaves invariant the space $L^2(\omega)$. Then by Theorem 9.1.5, $PV \subset V$. It follows from our assumptions that $\mu(\omega) = 0$ or $\mu(\Omega \setminus \omega) = 0$, which had to be shown. □

We will see that Theorem 10.1.5 gives a most convenient criterion for proving that elliptic operators generate irreducible positive semigroups.

## 10.2 Positive eigenfunctions

Our goal is to show that the generator of a positive irreducible semigroup has always a unique strictly positive eigenfunction whenever its resolvent is compact. Since this involves the spectrum of the operator we assume here that $\mathbb{K} = \mathbb{C}$.

Throughout this section we consider the complex Banach space $X = L^p(\Omega, \Sigma, \mu)$, $1 \le p \le \infty$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space. We include the case $p = \infty$ since we also want to consider the adjoint of a semigroup on $L^1(\Omega)$. Also for this reason we introduce resolvent positive operators. We frequently use the following.

**Lemma 10.2.1.** *A bounded operator $B$ on $L^p(\Omega)$ is positive if and only if $BL^p(\Omega,\mathbb{R}) \subset L^p(\Omega,\mathbb{R})$ and*

$$(10.1) \qquad |Bf| \le B|f| \qquad (f \in L^p(\Omega)).$$

*Proof.* In fact, assume that $B \ge 0$. Let $f \in L^p(\Omega)$. Then for $\theta \in \mathbb{R}$, $\mathrm{Re}(e^{i\theta} f) \le |f|$. Hence, $\mathrm{Re}(e^{i\theta} Bf) = B(\mathrm{Re}(e^{i\theta} f)) \le B|f|$. Since $\theta$ is arbitrary, it follows that $|Bf| \le B|f|$. ☐

**Definition 10.2.2.** *An operator $A$ on $X$ is called* **resolvent positive** *if there exists $\lambda_0 \in \mathbb{R}$ such that $(\lambda_0,\infty) \subset \rho(A)$ and $R(\lambda, A) \ge 0$ for all $\lambda > \lambda_0$.*

Recall that

$$s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$$

is the **spectral bound** of an operator $A$.

If $A$ generates a positive $C_0$-semigroup $(T(t))_{t\ge 0}$ on $L^p(\Omega)$, $1 \le p < \infty$, then $A$ is resolvent positive and $s(A) \le \omega(A)$. This follows from Proposition 2.4.1.

**Theorem 10.2.3.** *Let $A$ be a resolvent positive operator. If $s(A) > -\infty$, then $s(A) \in \sigma(A)$.*

*Proof.* 1. Let $s \in \mathbb{R}$ such that $(s,\infty) \subset \rho(A)$. We show that $\lambda \in \rho(A)$ whenever $\mathrm{Re}\lambda > s$. The proof is given in two steps.

a) We show that $R(\lambda, A) \ge 0$ for $\lambda > s$. In fact, let

$$\lambda_0 := \inf\{\mu > s : R(\lambda, A) \ge 0 \text{ for all } \lambda \ge \mu\}.$$

Assume that $\lambda_0 > s$. Since $R(\cdot, A)$ is continuous, it follows that $R(\lambda_0, A) \ge 0$. Consequently, $R(\lambda_0, A)^n \ge 0$ for all $n \in \mathbb{N}$. By Proposition 1.2.2 we obtain that

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1} \ge 0$$

for all $\lambda \in (\lambda_0 - \delta, \lambda_0]$ where $\delta = \|R(\lambda_0, A)\|^{-1}$. This contradicts the definition of $\lambda_0$.

b) By the resolvent identity, for $s < \lambda < \mu$,

$$(10.2) \qquad R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \ge 0.$$

Thus the function $R(\cdot, A)$ is decreasing on $(s, \infty)$.

c) We prove the claim. For this we may assume that $s = 0$ replacing $A$ by $A - s$ otherwise. Let $\lambda_0 > 0$. Let $r = \mathrm{dist}(\lambda_0, \sigma(A))$. We claim that $r \ge \lambda_0$. Since $\lambda_0 > 0$ may be chosen arbitrarily large, it then follows that the right-half plane is in $\rho(A)$. The power series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$$

of Proposition 1.2.2 converges whenever $|\lambda - \lambda_0| < r$. Now assume that $r < \lambda_0$. Then for $|\lambda - \lambda_0| < r$ we have for all $f \in X$

$$|R(\lambda, A)f| \le \sum_{n=0}^{\infty} |\lambda - \lambda_0|^n R(\lambda_0, A)^{n+1}|f| = R(\lambda_0 - |\lambda - \lambda_0|, A)|f| \le R(\lambda_0 - r, A)|f|,$$

by the monotonicity of the resolvent. Thus, the resolvent is bounded on the disk $D(\lambda_0, r) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$ by some constant $\rho > 0$. It follows from Proposition 1.2.2 that $\operatorname{dist}(\lambda, \sigma(A)) \ge \rho^{-1}$ for all $\lambda \in D(\lambda_0, r)$. Hence $r = \operatorname{dist}(\lambda_0, \sigma(A)) \ge r + \rho^{-1}$, a contradiction.

2. Let now $s_0 := \inf\{s \in \mathbb{R} : (s, \infty) \subset \rho(A)\}$. Then by 1. $s_0 = s(A)$. Thus, if $s_0 = -\infty$, then $\sigma(A) = \emptyset$ and so $s(A) = -\infty$. Assume that $s_0 > -\infty$. If $s_0 \in \rho(A)$, since $\rho(A)$ is open there exists $\delta > 0$ such that $(s_0 - \delta, \infty) \subset \rho(A)$, contradicting the definition of $s_0$. Thus $s_0 \in \sigma(A)$. ☐

**Corollary 10.2.4.** *Let $A$ be resolvent positive. Let $\lambda \in \rho(A)$ such that $R(\lambda, A) \ge 0$. Then $\lambda \in \mathbb{R}$ and $\lambda > s(A)$.*

*Proof.* Since $R(\lambda, A) \ge 0$ one has $R(\lambda, A)L^p(\Omega, \mathbb{R}) \subset L^p(\Omega, \mathbb{R})$. Let $f \in L^p(\Omega, \mathbb{R}) \setminus \{0\}$. Then $u = R(\lambda, A)f \in L^p(\Omega, \mathbb{R})$ and $\lambda u - Au = f$. Hence $\lambda u \in L^p(\Omega, \mathbb{R})$ and so $\lambda \in \mathbb{R}$. Assume that $\lambda < s(A)$. Then by the resolvent identity (10.2), $R(\lambda, A) \ge R(\mu, A) \ge 0$ for all $\mu > s(A)$. Hence $\|R(\mu, A)\| \le \|R(\lambda, A)\|$ for $\mu > s(A)$. On the other hand, since $s(A) \in \sigma(A)$, $\lim_{\mu \downarrow s(A)} \|R(\mu, A)\| = \infty$, by Proposition 1.2.2. ☐

Now we prove the existence of a positive eigenfunction. We will see that it is unique up to a scalar factor if $A$ generates an irreducible positive semigroup.

**Theorem 10.2.5** (Krein–Rutman)**.** *Let $A$ be a resolvent positive operator with compact resolvent. If $s(A) > -\infty$ then there exists $0 < u \in D(A)$ such that $Au = s(A)u$.*

*Proof.* We may assume that $s(A) = 0$ replacing $A$ by $A - s(A)$ otherwise. Since $0 \in \sigma(A)$ it follows from Proposition 1.2.2 that $\|R(\lambda_n, A)\| \to \infty$ as $n \to \infty$ for some $\lambda_n > 0$ with $\lim_{n\to\infty} \lambda_n = 0$. By the Uniform Boundedness Principle there exists $f \in X$ such that $\|R(\lambda_n, A)f\| \to \infty$ as $n \to \infty$. Since $|R(\lambda_n, A)f| \le R(\lambda_n, A)|f|$ we may assume that $f > 0$. Let $u_n = \|R(\lambda_n, A)f\|^{-1} R(\lambda_n, A)f$. Then $0 < u_n \in D(A)$, $\|u_n\| = 1$ and

$$\lambda_n u_n - Au_n = \|R(\lambda_n, A)f\|^{-1} f.$$

Thus $\lambda_n u_n - Au_n \to 0$ as $n \to \infty$. Hence $(u_n)_{n\in\mathbb{N}}$ is bounded in the graph norm. Since the embedding $D(A) \hookrightarrow X$ is compact we may assume that $u = \lim_{n\to\infty} u_n$ exists in $X$, considering a subsequence otherwise. Then $\|u\| = 1$, $u > 0$, and since $A$ is closed, $-Au = 0$. ☐

Now we consider $X = L^p(\Omega)$, $1 \le p < \infty$, and apply the preceding result to irreducible positive semigroups.

**Theorem 10.2.6.** *Let $A$ be the generator of a positive, irreducible $C_0$-semigroup $(T(t))_{t\ge 0}$. Assume that $A$ has compact resolvent. Assume that $s(A) = 0$. Then there exists $0 \ll u \in \ker(A)$ and $0 \ll \phi \in \ker(A')$. Moreover,* $\dim\ker(A) = \dim\ker(A') = 1$.

Note that for $f \in X$ one has $f \in \ker(A)$ if and only if $T(t)f = f$ $(t \ge 0)$ and for $\phi \in X'$ one has $\phi \in \ker(A')$ if and only if $T(t)'\phi = \phi$ $(t \ge 0)$. This follows from Proposition 2.2.4.

*Proof.* a) By the previous Theorem 10.2.5 there exists $0 < u \in \ker(A)$. We show that $u \gg 0$. Let $\omega = \{x \in \Omega : u(x) = 0\}$. Let $0 \le f \in L^p(\omega)$. Then $f = \lim_{k\to\infty} f_k$ where $f_k = f \wedge ku$. Since $T(t)f_k \le kT(t)u = ku$, one has $T(t)f_k \in L^p(\omega)$ for all $k \in \mathbb{N}$. Hence $T(t)f \in L^p(\omega)$ for all $t \ge 0$. Since $T$ is irreducible and $\mu(\Omega \setminus \omega) > 0$, we deduce that $\mu(\omega) = 0$.

b) Note that $A'$ is resolvent positive and $s(A') = s(A) = 0$. By Theorem 10.2.5 there exists $0 < \phi \in \ker A'$. We claim that $\phi \gg 0$. In fact, let $\omega = \{x \in \Omega : \phi(x) = 0\}$. Let $0 \le f \in L^p(\omega)$. Then $< T(t)f, \phi >=< f, T(t)'\phi >=< f, \phi >= 0$. Hence $T(t)f = 0$ on $\Omega \setminus \omega = \{x \in \Omega : \phi(x) > 0\}$, i.e., $T(t)f \in L^p(\omega)$. Since $\mu(\Omega \setminus \omega) > 0$, we deduce that $\mu(\omega) = 0$.

c) Let $f \in \ker(A)$. Since $T(t) \ge 0$ one has $f^+ = (T(t)f)^+ \le T(t)f^+$. Hence $T(t)f^+ - f^+ \ge 0$. Moreover, $< T(t)f^+ - f^+, \phi >=< f^+, T(t)'\phi > - < f^+, \phi >= 0$. Since $\phi \gg 0$, it follows that $T(t)f^+ = f^+$ for all $f \in \ker(A)$. Thus, $f \in \ker(A)$ implies $f^+, f^- \in \ker(A)$. Since $T$ is irreducible, we conclude from a) that $f^+ = 0$ or $f^+ \gg 0$ for each $f \in \ker(A)$. In particular, for each $f \in \ker(A)$ one has $f \ge 0$ or $f \le 0$. This implies that $\dim\ker(A) = 1$ by the following Lemma.

d) If $\psi \in \ker(A')$, then $\psi^+ = (T(t)'\psi)^+ \le T(t)'\psi^+$ and thus $T(t)'\psi^+ - \psi^+ \ge 0$, but $< T(t)'\psi^+ - \psi^+, u >= 0$. Hence $T(t)'\psi^+ = \psi^+$. Now the same proof as in c) shows that $\dim\ker(A') = 1$. □

**Lemma 10.2.7.** *Let $\mathbb{K} = \mathbb{R}$. Let $Y$ be a subspace of $L^q(\Omega)$, $1 \le q \le \infty$, such that $f \in Y$ implies $f \ge 0$ or $f \le 0$. Then* $\dim Y \le 1$.

*Proof.* Let $f_1, f_2 \in Y$ such that $f_1 \ne 0 \ne f_2$. We want to show that $f_1 = \lambda f_2$ for some $\lambda \in \mathbb{R}$. Since $f_2 > 0$ or $f_2 < 0$, we may assume that $f_2 > 0$ (replacing $f_2$ by $-f_2$ otherwise). Let

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} : f_1 \le \lambda f_2\}.$$

Then $\lambda_0 > -\infty$. In fact, otherwise, $f_1 \le -nf_2$ for all $n \in \mathbb{N}$. Hence $f_2 \le -\frac{1}{n} f_1$ for all $n \in \mathbb{N}$. Thus $f_2 \le 0$, which contradicts that $f_2 > 0$. Then $f_1 \le \lambda_0 f_2$ and $\lambda_0$ is minimal for this property. Since each two functions are compareable we conclude that $f_1 \ge (\lambda_0 - \frac{1}{n}) f_2$ for all $n \in \mathbb{N}$. Consequently, $f_1 \ge \lambda_0 f_2$. We have shown that $f_1 = \lambda_0 f_2$. □

We usually assume that $s(A) = 0$. This can be always obtained by rescaling. In fact, the following theorem shows that $s(A) > -\infty$ whenever $(e^{-tA})_{t\geq 0}$ is positive irreducible and $A$ has compact resolvent. Thus we may replace $A$ by $A - s(A)$ to have the convenient situation that the spectral bound is 0.

**Theorem 10.2.8** (de Pagter)**.** *Let $A$ be the generator of a positive, irreducible $C_0$-semigroup. Assume that $A$ has compact resolvent. Then $s(A) > -\infty$.*

We do not give a proof of this important and deep result.

Our last result will help us to locate the spectral bound in examples.

**Proposition 10.2.9.** *Let $(T(t))_{t\geq 0}$ be a positive, irreducible $C_0$-semigroup whose generator $A$ has compact resolvent. Let $0 < u \in D(A)$. Then the following assertions hold.*

*a) If $Au \leq 0$, then $s(A) \leq 0$;*

*b) if $Au < 0$, then $s(A) < 0$;*

*c) if $Au \geq 0$, then $s(A) \geq 0$;*

*d) if $Au > 0$, then $s(A) > 0$;*

*e) if $Au = 0$, then $s(A) = 0$.*

*Proof.* By de Pagter's theorem we have $s(A) > -\infty$. Applying Theorem 10.2.6 to $A - s(A)$, we find $0 \ll \phi \in D(A')$ such that $A'\phi = s(A)\phi$. Now let $0 < u \in D(A)$. Then

$$\langle Au, \phi\rangle = \langle u, A'\phi\rangle = s(A)\langle u, \phi\rangle.$$

Since $\langle u, \phi\rangle > 0$, the five assertions follow. ☐

A similar argument shows that the spectral bound is strictly increasing as a function of the generator. Also this is useful in order to locate the spectral bound in examples.

**Theorem 10.2.10.** *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be positive, irreducible semigroups whose generators $A$ and $B$ have compact resolvent. Assume that*

$$0 \leq S(t) \leq T(t) \qquad (t \geq 0).$$

*If $A \neq B$, then $s(A) < s(B)$.*

*Proof.* Assume that $s(A) = 0$. Then there exists $0 \ll u_1 \in \ker(A)$. Hence $u_1 = S(t)u_1 \leq T(t)u_1$ $(t \geq 0)$. If follows from Exercise 10.5.1 that $s(B) \geq 0$. Assume that $s(B) = 0$. Then there exists $0 \ll \phi_1 \in \ker(B')$. Then

$$\langle T(t)u_1 - u_1, \phi_1\rangle = \langle u_1, T(t)'\phi_1 - \phi_1\rangle = 0 \qquad (t \geq 0).$$

Since $T(t)u_1 - u_1 \geq 0$ and $\phi_1 \gg 0$, it follows that $T(t)u_1 = u_1$ $(t \geq 0)$. Observe that $T(t)'\phi_1 - S(t)'\phi_1 \geq 0$ because $S(t) \leq T(t)$. Since $T(t)u_1 = S(t)u_1 = u_1$, it follows that $\langle u_1, T(t)'\phi_1 - S(t)'\phi_1\rangle = 0$. Since $u_1 \gg 0$ we conclude that $T(t)'\phi_1 = S(t)'\phi_1$. Thus $\phi_1 = T(t)'\phi_1 = S(t)'\phi_1$ for all $t \geq 0$. Now let $0 \leq f \in X$. Then $T(t)f - S(t)f \geq 0$ and $\langle T(t)f - S(t)f, \phi_1\rangle = \langle f, T(t)'\phi_1 - S(t)'\phi_1\rangle = 0$. Since $\phi_1 \gg 0$, it follows that $S(t)f = T(t)f$ for all $t \geq 0$. $\square$

## 10.3 Dominant eigenvalues

We proceed by a more subtle analysis of the boundary spectrum $\sigma(A) \cap (s(A) + i\mathbb{R})$. Our goal is to prove that $s(A)$ is a dominant eigenvalue under suitable conditions.

Throughout this section $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space and we consider the complex spaces $L^p(\Omega)$, $1 \leq p \leq \infty$. We start by some auxiliary results on positive operators on $L^\infty$. First we recall that

$$|Bf| \leq B|f| \qquad (f \in L^\infty) \tag{10.3}$$

whenever $B \in \mathcal{L}(L^\infty)$ is positive, see Lemma 10.2.1.

**Proposition 10.3.1.** *Let $B$ be a bounded operator on $L^\infty(\Omega)$.*

*a) If $B \geq 0$, then $\|B\| = \|B1_\Omega\|_\infty$.*

*b) Assume that $B1_\Omega = 1_\Omega$. Then $B \geq 0$ if and only if $\|B\| \leq 1$.*

*Proof.* a) If $\|f\|_\infty \leq 1$, then $|f| \leq 1_\Omega$. Hence, $|Bf| \leq B|f| \leq B1_\Omega$ by (10.3). Hence $\|Bf\|_\infty \leq \|B1_\Omega\|_\infty$.

b) If $B \geq 0$, it follows from a) that $\|B\| = \|B1_\Omega\|_\infty = \|1_\Omega\|_\infty = 1$. Conversely, assume that $\|B\| = 1$. We first observe the following: Let $f \in L^\infty(\Omega)$. Then

$$\begin{gathered} f(x) \in [-1,1] \text{ for a.e. } x \in \Omega \text{ if and only if} \\ \|f + ir1_\Omega\|_\infty^2 \leq 1 + r^2 \text{ for all } r \in \mathbb{R}. \end{gathered} \tag{10.4}$$

In fact, assume that $\|f + ir1_\Omega\|_\infty^2 \leq 1 + r^2$ for all $r \in \mathbb{R}$. Then

$$\mathrm{Re} f(x)^2 + (r + \mathrm{Im} f(x))^2 = \mathrm{Re} f(x)^2 + \mathrm{Im} f(x)^2 + 2\mathrm{Im} f(x) r + r^2 \leq 1 + r^2 \text{ a.e.} \tag{10.5}$$

In particular $2\mathrm{Im} f(x) r \leq 1$ a.e. for all $r \in \mathbb{R}$. This implies that $\mathrm{Im} f(x) = 0$ for a.e. $x \in \Omega$. Thus $f$ is real, and (10.5) implies that $f(x) \in [-1,1]$ for a.e. $x \in \Omega$. The converse implication in (10.4) is clear.

In order to prove that $B \geq 0$ let $0 \leq f \leq 2 \cdot 1_\Omega$. Then $-1_\Omega \leq f - 1_\Omega \leq 1_\Omega$. Then by (10.4)

$$\|f - 1_\Omega + ir1_\Omega\|_\infty \leq 1 + r^2 \qquad (r \in \mathbb{R}).$$

Since $\|B\| \leq 1$, it follows that

$$\|Bf - 1_\Omega + ir1_\Omega\|_\infty = \|B(f - 1_\Omega + ir1_\Omega)\|_\infty \leq 1 + r^2 \qquad (r \in \mathbb{R})$$

for all $r \in \mathbb{R}$. Consequently, $-1_\Omega \leq Bf - 1_\Omega \leq 1_\Omega$, i.e., $0 \leq Bf \leq 2 \cdot 1_\Omega$. □

**Proposition 10.3.2.** *Let $B : L^\infty(\Omega) \to L^\infty(\Omega)$ be a positive operator such that $B1_\Omega = 1_\Omega$. Let $h \in L^\infty(\Omega)$, $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $Bh = \lambda h$, with $|h| = 1_\Omega$. Then $Bh^m = \lambda^m h^m$ for all $m \in \mathbb{Z}$.*

*Proof.* Let $Rf := \overline{\lambda h} B(fh)$. Then $R : L^\infty(\Omega) \to L^\infty(\Omega)$ is linear and by (10.3)

$$|Rf| \leq B|f| \qquad (f \in L^\infty(\Omega)). \tag{10.6}$$

Hence $\|R\| \leq 1$. Since $R1_\Omega = 1_\Omega$, it follows from Proposition 10.3.1(b) that $R \geq 0$. Then by (10.6) $B - R \geq 0$. Hence by Proposition 10.3.1(a) $\|B - R\| = \|(B - R)1_\Omega\|_\infty = 0$. Thus $R = B$. From the definition of $R$ we obtain that

$$B(fh) = \lambda h Bf \qquad (f \in L^\infty(\Omega)).$$

Applying this successively to $f = h, h^2, \ldots$ yields the claim. □

Now we can prove the main result of this section. Recall that we assume that $s(A) = 0$.

**Theorem 10.3.3** (cyclicity of the boundary spectrum)**.** *Let $(T(t))_{t\geq 0}$ be a positive irreducible $C_0$-semigroup on $L^p(\Omega)$ where $1 \leq p < \infty$. Assume that the generator $A$ of $(T(t))_{t\geq 0}$ has compact resolvent and that $s(A) = 0$. Let $\beta \in \mathbb{R}$. If $i\beta \in \sigma(A)$, then $i\beta m \in \sigma(A)$ for all $m \in \mathbb{Z}$.*

*Proof.* Let $h \in D(A)$, $h \neq 0$ such that $Ah = i\beta h$. Note that $A - i\beta$ generates the semigroup $(e^{-i\beta t}T(t))_{t\geq 0}$. Thus it follows from (2.9) that

$$e^{-i\beta t}T(t)h - h = \int_0^t e^{-i\beta s}T(s)(A - i\beta)h\,ds = 0.$$

Hence $T(t)h = e^{i\beta t}h$ $(t \geq 0)$. Consequently, $|h| = |T(t)h| \leq T(t)|h|$ $(t \geq 0)$. There exists $0 \ll \phi \in \ker(A')$. Hence $T(t)'\phi = \phi$ $(t \geq 0)$. Thus

$$\langle T(t)|h| - |h|, \phi\rangle = \langle |h|, T(t)'\phi - \phi\rangle = 0.$$

Since $T(t)|h| - |h| \geq 0$ this implies that $T(t)|h| = |h|$ for all $t \geq 0$. Thus $0 < u := |h| \in \ker(A)$. Consequently $u \gg 0$.

The following argument shows that we can assume that $\mu(\Omega) < \infty$ and $u = 1_\Omega$. Consider the isomorphism

$$\Phi : L^p(\Omega, \mu) \ni f \mapsto \frac{f}{u} \in L^p(\Omega, u^p\mu)$$

and the semigroup $(S(t))_{t\geq 0}$ given by

$$S(t)f := \Phi T(t)\Phi^{-1} f = \frac{1}{u} T(t)(uf).$$

Then $(S(t))_{t\geq 0}$ is positive and irreducible. Moreover, $S(t)1_\Omega = 1_\Omega$ $(t \geq 0)$. Replacing $(T(t))_{t\geq 0}$ by $(S(t))_{t\geq 0}$ and $L^p(\Omega, \mu)$ by $L^p(\Omega, u^p\mu)$ we may assume that $\mu(\Omega) < \infty$ and $u = 1_\Omega$, which we do now.

Then $|h| = 1$ and $T(t)h = e^{i\beta t}h$ $(t \geq 0)$. It follows from Proposition 10.3.2 that $T(t)h^m = e^{i\beta m t}h^m$. Hence $Ah^m = i\beta m h^m$ for all $m \in \mathbb{Z}$. □

If in Theorem 10.3.3 the semigroup $(T(t))_{t\geq 0}$ is holomorphic, then $\sigma(A) \cap i\mathbb{R} = \{0\}$. and 0 is a dominant eigenvalue.

**Corollary 10.3.4** (dominant eigenvalue)**.** *Let $(T(t))_{t\geq 0}$ be a holomorphic, positive, irreducible $C_0$-semigroup on $L^p(\Omega)$, $1 \leq p < \infty$, whose generator has compact resolvent. Assume that $s(A) = 0$. Then there exists $\delta > 0$ such that*

$$\sigma(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\delta\} = \{0\}. \tag{10.7}$$

*Proof.* Since $A$ generates a holomorphic semigroup, there exist $\omega \in \mathbb{R}$ and $\alpha \in (\frac{\pi}{2}, \pi)$ such that

$$\sigma(A) \subset \omega + \Sigma_\alpha. \tag{10.8}$$

This follows from Theorem 2.5.3. In particular, $\sigma(A) \cap i\mathbb{R}$ is bounded. Thus, it follows from Theorem 10.3.3 that $\sigma(A) \cap i\mathbb{R} = \{0\}$. Now assume that there exists $\lambda_n \in \sigma(A)$ such that $\mathrm{Re}\lambda_n < 0$ and $\lim_{n\to\infty} \mathrm{Re}\lambda_n = 0$. Then by (10.8) $(\lambda_n)_{n\in\mathbb{N}}$ is bounded. Hence $(\lambda_n)_{n\in\mathbb{N}}$ has a convergent subsequence. This contradicts the fact that $\sigma(A)$ consists merely of isolated points. □

## 10.4 Asymptotic behaviour

Now we can use the preceding results to prove that an irreducible positive holomorphic $C_0$-semigroup converges to an equilibrium if the resolvent is compact and if the semigroup is normalized in such a way that the spectral bound of its generator is 0.

**Theorem 10.4.1** (convergence to an equilibrium)**.** *Let $(T(t))_{t\geq 0}$ be a holomorphic, positive, irreducible $C_0$-semigroup on $L^p(\Omega)$, where $1 \leq p < \infty$, whose generator $A$ has compact resolvent. Assume that $s(A) = 0$. Then $(T(t))_{t\geq 0}$ converges to a projection $P$ in $\mathcal{L}(L^p(\Omega))$ as $t \to \infty$. The projection $P$ is of the form*

$$Pf = \langle f, \phi \rangle u,$$

*where $0 \ll u \in \ker(A)$, $0 \ll \phi \in \ker(A')$, $\langle u, \phi \rangle = 1$.*

*Proof.* By Theorem 10.2.6 there exist $0 \ll u \in \ker(A)$ and $0 \ll \phi \in \ker(A')$. Replacing $u$ by a scalar multiple we may assume that $\langle u, \phi \rangle = 1$. Then $Pf = \langle f, \phi \rangle u$ defines a projection $P \in \mathcal{L}(L^p(\Omega))$. Since $T(t)u = u$ and $T(t)'\phi = \phi$ for all $t \geq 0$, it follows that

$$T(t)P = PT(t) = P \qquad (t \geq 0). \tag{10.9}$$

Then the spaces $X_1 := PX$ and $X_0 := \ker(P)$ are invariant under the semigroup.

Let $T_0(t) := T(t)_{|X_0}$. Then $(T_0(t))_{t\geq 0}$ is a holomorphic semigroup whose generator we denote by $A_0$. Let $\lambda \in \rho(A)$. Then $\lambda R(\lambda, A)P = P\lambda R(\lambda, A) = P$. Hence $R(\lambda, A)X_0 \subset X_0$. It follows that $\lambda \in \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)_{|X_0}$ (cf. Exercise 10.5.3). We have shown that $\rho(A) \subset \rho(A_0)$. Next we show that $0 \notin \sigma(A_0)$. Otherwise, there exists $0 \neq v \in \ker(A_0)$. Then $v \in \ker(A)$. Since $\dim \ker(A) = 1$ it follows that $v = cu$ for some $c \in \mathbb{K}$. Hence $v = 0$ since $v \in X_0$, a contradiction. We have shown that $0 \in \rho(A_0)$. Since 0 is dominant, it follows that $s(A_0) < 0$. But $s(A_0) = \omega(A_0)$ since $T_0$ is holomorphic (see [ABHN01, Theorem 5.1.12] and [EN00, Cor. 3.12]). Consequently, letting $s(A_0) < -\delta < 0$ we find $M \geq 1$ such that $\|T_0(t)\| \leq Me^{-\delta t}$ $(t \geq 0)$. Thus

$$\begin{aligned} \|T(t) - P\| &= \|T(t)P - P + T(t)(I - P)\| \\ &= \|T(t)(I - P)\| \\ &\leq Me^{-\delta t} \qquad (t \geq 0). \end{aligned}$$

This concludes the proof. □

## 10.5 Exercises

Throughout this Exercise section $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space. In the first exercise we suggest a generalisation of Proposition 10.2.9.

**Exercise 10.5.1** (Estimates of $s(A)$)**.** *Let $(T(t))_{t\geq 0}$ be a positive, irreducible $C_0$-semigroup on $L^p(\Omega)$, $1 \leq p < \infty$ whose generator $A$ has compact resolvent.*

1. *Let $0 < u \in L^p(\Omega)$. Show the following.*
   *(a) If $T(t)u \leq u$ for some $t > 0$, then $s(A) \leq 0$.*
   *(b) If $T(t)u \geq u$ for some $t > 0$, then $s(A) \geq 0$.*

*2. Let $0 < \phi \in L^p(\Omega)'$. Show the following.*

*(c) If $T(t)'\phi \le \phi$ for all $t \ge 0$, then $s(A) \le 0$.*

*(Hint: Consider $V := \{f \in L^p(\Omega) : < |f|, \phi >= 0\}$. Is $T(t)V \subset V$? Is $\phi \gg 0$?)*

*(d) If $T(t)'\phi \ge \phi$ for all $t \ge 0$, then $s(A) \ge 0$.*

We now use Exercise 10.5.1 to find a criterion for forms.

**Exercise 10.5.2** (Estimates for $s(A)$ in the form case)**.** *Assume that $(T(t))_{t\ge0}$ is an irreducible, positive $C_0$-semigroup whose generator $-A$ has compact resolvent. Assume that $p = 2$ and that $A$ is associated with a continuous, densely defined elliptic form $(a, V)$.*

1. *Assume that there exists $0 < u \in V$ such that $a(u, v) \ge 0$ for all $v \in V_+$. Show that $s(A) \le 0$. (Hint: use Exercise 10.5.1(a).)*

2. *Formulate and prove criteria that are similar to (a) and corresponding to the assertions of Exercise 10.5.1(b)–(d).*

Next we investigate how the invariant spaces influence the spectrum. This was used in the proof of Theorem 10.4.1.

**Exercise 10.5.3** (spectrum and invariant spaces)**.** *Let $A$ be an operator on a Banach space $X$.*

1. *Let $Y$ be a Banach space such that $Y \hookrightarrow X$. Assume that there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A)Y \subset Y$. Let $\lambda \in \rho(A)$.*

   *(a) Let $\lambda$ be in the component $\mathcal{C}$ of $\rho(A)$ containing $\lambda_0$. Show that $R(\lambda, A)Y \subset Y$.*

   *(b) Define the part $A_Y$ of $A$ in $Y$ by*

$$\begin{aligned} D(A_Y) &:= \{x \in D(A) \cap Y : Ay \in Y\} \\ A_Y y &:= Ay. \end{aligned}$$

   *Show that $\mathcal{C} \subset \rho(A_Y)$ and $R(\lambda, A_Y) = R(\lambda, A)_{|Y}$ for all $\lambda \in \mathcal{C}$.*

2. *Let $Y$ be a Banach space such that $D(A) \subset Y \hookrightarrow X$. Assume that $\rho(A) \ne 0$. Show that $\rho(A_Y) = \rho(A)$.*

3. *Consider Theorem 7.1.5. Show that $\sigma(A) = \sigma(\mathcal{A})$.*

Finally, we show that the strong limit of an irreducible semigroup is necessarily of rank $\le 1$.

**Exercise 10.5.4** (Rank-1 projection)**.** *Let $(T(t))_{t\ge0}$ be a positive, irreducible $C_0$-semigroup on $X = L^p(\Omega)$, where $1 \le p < \infty$, with generator $A$. Assume that $Pf = \lim_{t\to\infty} T(t)f$ exists for all $f \in X$. Show that $P = 0$ or $P$ is given by $Pf = \langle f, \phi\rangle u$ $(f \in X)$ where $0 \ll u \in \ker(A)$, $0 \ll \phi \in D(A')$.*

Motivated by Theorem 10.1.2, we add an open problem to which we do not know the answer yet.

**Open Problem 10.5.5.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $k \in L^\infty(\Omega \times \Omega)$. Consider the kernel operator $B$ on $L^p(\Omega)$ given by*

$$Bf(x) := \int_\Omega k(x,y)f(y)dy \qquad (f \in L^p(\Omega),\ x \in \Omega).$$

*Assume that*

$$Bf(x) > 0 \qquad \text{for a.e. } x \in \Omega$$

*whenever $0 < f \in L^p(\Omega)$. Does it follow that $k(x,y) > 0$ for a.e. $(x,y) \in \Omega \times \Omega$?*

**Answer** (by Tomás Mátrai, Budapest):

Let $\lambda_m$ denote the $m$-dimensional outer Lebesgue measure on $\mathbb{R}^m$ and let $\mu$ stand for an arbitrary Hausdorff measure on the space where it is defined. The counterexample is based on the following result

**Theorem ( [Buc94, Theorem 1])**
For every $\varepsilon > 0$ there exists a Lebesgue measurable set $E \subset [0,1] \times [0,1]$ such that $\lambda_2(E) > 1 - \varepsilon$ and whenever $A \times B \subset E$ then either $\mu(A) = 0$ or $\lambda_1(B) = 0$.

By inner compact regularity of the Lebesgue measure we can assume that $E$ is in fact compact. Set $\Omega = (0,1)$, $k(x,y) = 1 - \chi_E(x,y)$, i.e. $k(x,y) = 0$ iff $(x,y) \in E$. Thus $k(x,y) > 0$ a.e. fails.

Let $0 < f$ be Lebesgue measurable. Again by inner compact regularity there is a $\delta > 0$ and a compact set $B \subset (0,1)$ such that $\lambda_1(B) > 0$ and $f(y) > \delta$ $(y \in B)$. By throwing away certain portions of $B$ we can assume that for every open set $U \subset (0,1)$,

$$U \cap B \neq \emptyset \quad \text{implies} \quad \lambda_1(U \cap B) > 0. \tag{10.10}$$

Suppose that for an $x \in (0,1)$,

$$(\mathcal{B}f)(x) = \int_0^1 k(x,y)f(y)\, dy = 0.$$

By (10.10) this implies $\{x\} \times B \subset E$. Thus for $A = \{x \in (0,1) : (\mathcal{B}f)(x) = 0\}$ we have $A \times B \subset E$, so by the special choice of $E$ we have $\mu(A) = 0$. For $\mu = \lambda_1$ this gives $\mathcal{B}f(x) > 0$ a.e., as required. □

## 10.6 Comments

This lecture contains an introduction to Perron–Frobenius Theory, reduced to some essential points. We give several comments.

### 10.6.1 Perron–Frobenius Theory

O. Perron and G. Frobenius developped a theory of positive matrices early last century (1907–1912), see Chapter 1 of Schaefer's monography [Sch74] for a beautiful presentation of this theory. The notion of irreducibility was implicit in this early work, but it was Schaefer who gave the general definition of positive irreducible operators on Banach lattices in 1960. He and his school in Tübingen later developped a complete theory in infinite dimensional spaces, see [Sch74]. The concept of irreducibility turned out to be most fruitful. The general definition is as follows.

A subspace $J$ of a Banach lattice $X$ is called an **ideal** if for $u, v \in X$

$$|v| \leq |u| \text{ and } u \in J \text{ imply that } v \in J.$$

A positive operator $B$ on $X$ is called **irreducible** if no non-trivial closed ideal is invariant under $B$.

This definition corresponds to ours (which is formulated for semigroups instead of bounded operators), since the closed ideals of $L^p(\Omega)$ are exactly the spaces $L^p(\omega)$ for $\omega \in \Sigma$ if $1 \leq p < \infty$.

If $B$ is a positive compact operator and the spectral radius $r(B)$ is $> 0$, then there exists a positive eigenfunction corresponding to $r(B)$. This is the classical Krein–Rutman Theorem. However, it can happen that the spectral radius is 0. It was Schaefer's conjecture that $r(B) > 0$ whenever $B$ is compact and irreducible. This problem is relatively easy to solve on some special spaces (as $L^1(\Omega)$ or $C(\overline{\Omega})$) but for the general case, the important $L^p$-case included, it turned out to be very hard. Finally, it was B. de Pagter who solved the problem in 1986, see [dPa86].

**Theorem 10.6.1** (de Pagter)**.** *Every positive, irreducible compact operator on a Banach lattice has a positive spectral radius.*

De Pagter used in a sophisticated way the technique of Lomonosov introduced for the invariant subspace problem. Applying de Pagter's result to the resolvent one actually obtains Theorem 10.2.8.

Other crucial results of the Perron–Frobenius theory concern the boundary spectrum. If $B$ is a compact positive operator and $r(B) > 0$ then, if $\lambda$ is an eigenvalue of modulus $r(B)$, also $\lambda^m$ is an eigenvalue for all $m \in \mathbb{Z}$. This means that the spectrum is **cyclic**. For matrices this is a result of Perron–Frobenius.

The systematic development of Perron–Frobenius Theory for positive semigroups started around 1980 (see [Nag86]). Theorem 10.3.3 is due to G. Greiner. It is valid in arbitrary Banach lattices. Our simple similarity argument in the proof avoids the use of Kakutani's Theorem. More generally, the following cyclicity result holds for the boundary spectrum (and not just the boundary *point* spectrum).

**Theorem 10.6.2** (Greiner)**.** *Let $A$ be the generator of a bounded, positive $C_0$-semigroup on a complex Banach lattice. Let $\beta \in \mathbb{R}$. If $i\beta \in \sigma(A)$, then $im\beta \in \sigma(A)$ for all $m \in \mathbb{Z}$.*

Again, if the semigroup generated by $A$ is holomorphic (or merely eventually norm continuous), one deduces that

$$\sigma(A) \cap i\mathbb{R} \subset \{0\}.$$

If $X$ is reflexive, this implies that

$$P = \lim_{t \to \infty} T(t)$$

in the strong sense, by the Arendt–Batty–Lyubich–Vu Theorem (see [ABHN01, Theorem 5.5.6] or [EN00, Theorem V.2.21]).

Now we give further comments on the diverse sections.

### 10.6.2 Irreducibility and holomorphy

Theorem 10.1.2 is due to Majewski–Robinson [MR83], see also [Nag86, Chapter III] and the comments given there. The simple and most useful irreducibility criterion for semigroups associated with a form is due to Ouhabaz (see [Ouh05] and the references given there).

### 10.6.3 More compactness and irreducibility

Theorem 10.2.3 is valid on arbitrary ordered Banach spaces with normal cone, see [ABHN01, Proposition 3.11.2]. Theorems 10.2.5–10.2.6 are of Krein–Rutman type. They are valid on each complex Banach lattice. For more information concerning Proposition 10.2.9 and Theorem 10.2.10 see [Nag86]. In Proposition 10.2.8 it is not necessary to assume compactness of the lower semigroup. The following is a consequence of results by Fremlin–Dodds and Aliprantis–Burkinshaw. We refer to [MN91] for these results on positive operators, where also a proof of de Pagter's Theorem is given.

**Theorem 10.6.3.** *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be $C_0$-semigroups on a Banach lattice $X$ with generators $A$ and $B$. Assume that*

$$0 \leq S(t) \leq T(t) \qquad (t \geq 0).$$

*If $B$ has compact resolvent, then $A$ has compact resolvent.*

### 10.6.4 Semigroups on $L^\infty(\Omega)$, why not?

By a result of Lotz (see [ABHN01, p. 275]) each $C_0$-semigroup on $L^\infty$ has a bounded generator. Concerning irreducibility one should be aware that in $L^\infty(\Omega)$ there are many more closed ideals than those which are of the form $L^\infty(\omega)$, $\omega \in \Sigma$.

# Lecture 11

# Elliptic Operators and Domination

In this lecture we consider general elliptic operators with measurable coefficients. We prove irreducibility and formulate some of the consequences. Special attention is given to operators with unbounded drift term. For those, compensation by the term of order 0, i.e. the absorption term, is needed. Finally we will give a criterion which allows us to prove domination of a positive semigroup by another one. As applicaton we consider an elliptic operator with unbounded drift which is not associated to a form. The corresponding semigroup will be approximated by nicer semigroups from below. There are four sections.

- Irreducibility of semigrous generated by the Dirichlet and Neumann Laplacian.
- General elliptic operators.
- Domination.
- Approximation from below.

## 11.1 Irreducibility of the semigroups generated by the Dirichlet and Neumann Laplacian

In order to apply the criterion for irreducibility, Theorem 10.1.5, we need the following property of $H^1$.

**Lemma 11.1.1.** *Let $\Omega \subset \mathbb{R}^n$ be open and connected. If $\omega \subset \Omega$ is a Borel set such that $|\omega| > 0$ and $|\Omega \setminus \omega| > 0$, then there exist a ball $B = B(x_0, r_0) \subset \Omega$ and a test function $u \in \mathcal{D}(B)$ such that $1_\omega u \notin H^1(B)$.*

*Proof.* a) We show that there exists $x_0 \in \Omega$ such that $|\omega \cap B(x_0, r)| > 0$ and $|B(x_0, r) \setminus \omega| > 0$ for all $r > 0$. In fact, otherwise $\Omega = \Omega_1 \cup \Omega_2$ where

$$\begin{aligned}\Omega_1 &:= \{x \in \Omega : \exists\, r > 0 \text{ such that } B(x,r) \subset \Omega \text{ and } |B(x,r) \cap \omega| = 0\} \text{ and}\\ \Omega_2 &:= \{x \in \Omega : \exists\, r > 0 \text{ such that } B(x,r) \subset \Omega \text{ and } |B(x,r) \setminus \omega| = 0\}\ .\end{aligned}$$

Since $\Omega_1$ and $\Omega_2$ are open and $\Omega_1 \cap \Omega_2 = \emptyset$, it follows that $\Omega_1 = \Omega$ or $\Omega_2 = \Omega$. Assume that $\Omega_1 = \Omega$. Let $K \subset \omega$ be compact. For each $x \in K$ there exists a ball $B(x, r_x) \subset \Omega$ such that $|B(x, r_x) \cap \omega| = 0$. Since we can cover $K$ by a finite number of these balls, it follows that $|K| = 0$. Similarly, if $\Omega_2 = \Omega$ one obtains that $|\Omega \setminus \omega| = 0$. Both cases are contradictory to the assumption. This proves the claim.

b) Let $x_0 \in \Omega$ be the point of a) and let $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset \Omega$. Let $B = B(x_0, r_0)$ and let $u \in \mathcal{D}(B)$ such that $u(x_0) = 1$. Suppose that $1_\omega u \in H^1(B)$. It follows from Stampacchia's Lemma (Corollary 3.2.2) that $D_j(1_\omega u) = 1_\omega D_j u$ a.e.. This shows that $D_j(1_\omega u) \in L^\infty(B)$, $j = 1, \ldots, n$. Hence $1_\omega u \in W^{1,\infty}(B)$. It follows that there exists a continuous function $v : B \to \mathbb{R}$ such that $1_w u = v$ a.e. (see e.g. [Eva98, 5.8 Theorem 4]). By a) for each $k \in \mathbb{N}$, there exist $x_k, y_k \in B(x_0, 1/k)$ such that $|v(x_k) - v(y_k)| = |u(x_k)| \to 1 \quad (k \to \infty)$. But $x_k, y_k \to x_0$ as $k \to \infty$. Thus $v$ is not continuous at $x_0$. ∎

We will frequently use the following consequence of Lemma 11.1.1.

**Proposition 11.1.2.** *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Let $V$ be a subspace of $H^1(\Omega)$ containing $\mathcal{D}(\Omega)$. If $\omega \subset \Omega$ is a Borel set in $\Omega$ such that*

$$1_\omega V \subset V \ ,$$

*then $|\omega| = 0$ or $|\Omega \setminus \omega| = 0$.*

As a first example we consider the Dirichlet Laplacian.

**Example 11.1.3** (irreducibility of $(e^{t\Delta_\Omega^D})_{t\geq 0}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Denote by $\Delta_\Omega^D$ the Dirichlet Laplacian on $L^2(\Omega)$. Then the semigroup $(e^{t\Delta_\Omega^D})_{t\geq 0}$ is positive and irreducible. Assume in addition that $\Omega$ is bounded. Then there exists $0 \ll u \in D(\Delta_\Omega^D)$ such that*

$$-\Delta_\Omega^D u = \lambda_1 u$$

*where $\lambda_1 = -s(\Delta_\Omega^D)$ is the first eigenvalue of $-\Delta_\Omega^D$. Moreover,*

$$\dim(\ker(\lambda_1 + \Delta_\Omega^D)) = 1 \ .$$

*Proof.* The form domain of $\Delta_\Omega^D$ is $V = H_0^1(\Omega)$. It follows from Theorem 10.1.5 and the preceding proposition that $(e^{t\Delta_\Omega^D})_{t\geq 0}$ is irreducible. The semigroup $(e^{t\Delta_\Omega^D})_{t\geq 0}$ is holomorphic by Exercise 2.6.3. Moreover, $\Delta_\Omega^D$ has compact resolvent by Corollary 4.2.5. The last two assertions of the theorem follow now from Theorem 10.2.6. ∎

In order to establish compactness of the resolvent of the Neumann Laplacian we use the extension property.

**Definition 11.1.4** (extension property). *An open set $\Omega \subset \mathbb{R}^n$ has the* **extension property** *if for each $u \in H^1(\Omega)$ there exists $v \in H^1(\mathbb{R}^n)$ such that*

$$v_{|\Omega} = u \ .$$

*For example, if $\Omega$ is bounded and has Lipschitz boundary, then $\Omega$ has the extension property (see [Nec67], for the case where $\Omega$ is $C^1$ see also [Eva98] or [Bre83]).*

**Proposition 11.1.5** (extension operator). *Let $\Omega$ be a bounded open set with the extension property. Let $\tilde{\Omega}$ be an open set such that $\bar{\Omega} \subset \tilde{\Omega}$. Then there exists an operator $E \in \mathcal{L}(H^1(\Omega), H^1_0(\tilde{\Omega}))$ such that*

$$(Eu)_{|\Omega} = u$$

*for all $u \in H^1(\Omega)$. Such $E$ is called an* **extension operator**.

*Proof.* Let $\psi \in \mathcal{D}(\tilde{\Omega})$ such that $\psi_{|\Omega} \equiv 1$. For $u \in H^1(\Omega)$ there exists $v \in H^1(\mathbb{R}^n)$ such that $v_{|\Omega} = u$. Let $\omega = \psi v$. Then $\omega \in H^1_0(\tilde{\Omega})$ (by Proposition 3.2.8) and $\omega_{|\Omega} = u$. Thus the restriction operator

$$R : H^1_0(\tilde{\Omega}) \ni u \mapsto u_{|\Omega} \in H^1(\Omega)$$

is surjective. Denote by $R_0$ the restriction of $R$ to $(\ker R)^{\perp}$, the orthogonal complement of $\ker R$ in the Hilbert space $H^1_0(\tilde{\Omega})$. Then $R_0$ is an isomorphism. The mapping $E = R_0^{-1}$ is an extension operator. □

**Corollary 11.1.6.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set having the extension property. Then the injection*

$$H^1(\Omega) \hookrightarrow L^2(\Omega)$$

*is compact.*

*Proof.* Let $\tilde{\Omega}$ be bounded and open such that $\bar{\Omega} \subset \tilde{\Omega}$. Consider an extension operator $E : H^1(\Omega) \to H^1_0(\tilde{\Omega})$. Let $(u_k)$ be a bounded sequence in $H^1(\Omega)$. Then $(Eu_k)$ is bounded in $H^1_0(\tilde{\Omega})$. Since the injection $H^1_0(\tilde{\Omega})$ into $L^2(\tilde{\Omega})$ is compact (see Corollary 8.1.9), there exists a subsequence such that $Eu_{k_\ell}$ converges in $L^2(\tilde{\Omega})$ as $\ell \to \infty$. Hence $u_{k_\ell} = Eu_{k_\ell}|_\Omega$ converges in $L^2(\Omega)$ as $\ell \to \infty$. □

**Example 11.1.7.** *Let $\Omega \subset \mathbb{R}^n$ be an open, connected set. Consider the Neumann Laplacian $\Delta^N_\Omega$ on $L^2(\Omega)$. Then the semigroup $(e^{t\Delta^N_\Omega})_{t\geq 0}$ is positive and irreducible. If $\Omega$ is in addition bounded and has the extension property, then the following properties hold.*

*a) $e^{t\Delta^N_\Omega}$ is compact for all $t > 0$.*

*b) $\lim\limits_{t\to\infty} e^{t\Delta^N_\Omega} = P$ in $\mathcal{L}(L^2(\Omega))$, where $Pf = \frac{1}{|\Omega|}\int\limits_\Omega f dx \cdot 1_\Omega$.*

*Proof.* Irreducibility follows from Proposition 11.1.2, since the form domain is $V = H^1(\Omega)$. It follows from Corollary 11.1.6 that $\Delta_\Omega^N$ has compact resolvent. Since the semigroup $(e^{t\Delta_\Omega^N})_{t\geq 0}$ is holomorphic, it follows that $e^{t\Delta_\Omega^N}$ is compact for all $t > 0$ by Proposition 2.5.7. Observe that $1_\Omega \in D(\Delta_\Omega^N)$ and $\Delta_\Omega^N 1_\Omega = 0$. Hence $s(\Delta_\Omega^N) = 0$ by Proposition 10.2.9. By Theorem 10.4.1, $e^{t\Delta_\Omega^N}$ converges to a projection $P$ as $t \to \infty$ where $Pf = \langle f, \varphi\rangle u$ with $0 \ll u \in \ker(\Delta_\Omega^N)$. Thus $u \in H^1(\Omega)$ and $\int \nabla u \nabla v = 0$ for all $v \in H^1(\Omega)$. Hence $\Delta u = 0$, and consequently, $u \in C^\infty(\Omega)$. Taking $v = u$ we obtain $\int |\nabla u|^2 = 0$ and so $\nabla u = 0$ on $\Omega$. Since $\Omega$ is connected, this implies that $u$ is constant. Since $\Delta_\Omega^N$ is selfadjoint and $\dim(\ker \Delta_\Omega^N) = 1$, it follows that $\varphi = c \cdot u$ for some $c > 0$. This shows that $P$ has the desired form. $\square$

## 11.2 Elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be open. We define diverse realizations of an elliptic operator in the real space $H = L^2(\Omega)$. Let $a_{ij}, b_j, c_j, c_0 \in L^\infty(\Omega)$ be real coefficients, $i, j = 1, \ldots, n$, such that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \alpha\, |\xi|^2 \qquad (\xi \in \mathbb{R}^N)$$

for almost all $x \in \Omega$ and some $\alpha > 0$. Define

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

by

$$a(u,v) := \int_\Omega \left( \sum_{i,j=1}^{n} a_{ij} D_i u D_j v + \sum_{i=1}^{n} (b_i D_i u v + c_i u D_i v) + c_0 u v \right) dx. \tag{11.1}$$

Then $a$ is continuous and $H$-elliptic. Moreover,

$$a(u^+, u^-) = 0 \quad (u \in H^1(\Omega)) \; . \tag{11.2}$$

Now let $V$ be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Assume that

$$u \in V \quad \text{implies} \quad u^+ \in V \; . \tag{11.3}$$

Let $a_V$ be the restriction of $a$ to $V$. Denote by $A_V$ the operator associated with $a_V$.

**Theorem 11.2.1.** *Assume that $\Omega$ is connected. Then the semigroup $(e^{-tA_V})_{t\geq 0}$ generated by $-A_V$ on $L^2(\Omega)$ is positive and irreducible.*

*Proof.* Positivity follows from the first Beurling–Deny criterion (Theorem 9.2.1). Lemma 11.1.1 shows that for a Borel set $\omega \subset \Omega$ one has $1_\omega \cdot V \subset V$ only if $|\omega| = 0$ or $|\Omega \setminus \omega| = 0$. Hence $(e^{-tA_V})_{t\geq 0}$ is irreducible by Theorem 10.1.5. □

The operator $A_V$ is a realization of an elliptic operator with boundary conditions which are incorporated into $V$. We want to make this more precise. Define the elliptic operator $\mathcal{A} : H^1(\Omega) \to \mathcal{D}(\Omega)'$ by

$$\mathcal{A}u := -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + \sum_{j=1}^{n} b_jD_ju - \sum_{j=1}^{n} D_j(c_ju) + c_0u. \tag{11.4}$$

Here the distribution $D_i(a_{ij}D_ju)$ is defined by

$$\langle D_i(a_{ij}D_ju), v\rangle = -\int_\Omega a_{ij}D_juD_ivdx$$

for $v \in \mathcal{D}(\Omega)$. If $V = H_0^1(\Omega)$, then we call $A_V$ the **realization of $\mathcal{A}$ with Dirichlet boundary conditions**. Then

$$D(A_V) = \{u \in H_0^1(\Omega) : \mathcal{A}u \in L^2(\Omega)\}\ .$$

If $V = H^1(\Omega)$, then the domain of $A_V$ consists of functions which satisfy some kind of Neumann boundary conditions. We call $A_V$ the **realization of $\mathcal{A}$ with Neumann boundary conditions** even though these boundary conditions depend on the coefficients. We may also consider mixed boundary conditions. For example, if $\Omega$ is of the form $\tilde{\Omega} \setminus K$ where $\tilde{\Omega}$ is a bounded open set and $K \subset \tilde{\Omega}$ is compact, then $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \partial\tilde{\Omega}$ is the "exterior" and $\Gamma_2 = \partial K$ the "interior" boundary. If we take $V = \{u_{|\Omega} : u \in H_0^1(\tilde{\Omega})\}$ then this realizes Dirichlet boundary conditions on the exterior boundary and Neumann boundary conditions on the interior boundary.

In the following example we investigate the existence of an equilibrium.

**Example 11.2.2** (asymptotics for Neumann boundary conditions)**.** *Assume that $\Omega$ is bounded, connected and has the extension property. Let $V = H^1(\Omega)$. Assume that $c_0 = 0$ and*

*(a) $c_j = 0$ for $j = 1, \ldots, n$, or*

*(b) $b_j = 0$ for $j = 1, \ldots, n$.*

*Then $(e^{-tA_V})_{t\geq 0}$ converges in $\mathcal{L}(L^2(\Omega))$ to a rank-1 projection as $t \to \infty$.*

*Proof.* In the case (a), one has $a_V(1_\Omega, v) = 0$ for all $v \in V$. Hence $1_\Omega \in D(A_V)$ and $A_V1_\Omega = 0$. It follows from Proposition 10.2.9 that $s(A_V) = 0$ and the proof can be continued as in Example 11.1.7. In the case (b) apply case (a) to the adjoint of $A_V$. □

## 11.3 Domination

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, $H = L^2(\Omega)$, $V = H^1(\Omega)$. Let $(a_1, V_1)$ and $(a_2, V_2)$ be two continuous elliptic forms with associated operators $A_1$ and $A_2$. We assume that the semigroups $(e^{-tA_1})_{t\geq 0}$ and $(e^{-tA_2})_{t\geq 0}$ are positive. By the first Beurling–Deny criterion (Theorem 9.2.1), this means that

$$u \in V_j \quad \text{implies} \quad u^+ \in V_j \text{ and } a_j(u^+, u^-) \leq 0 \qquad (j = 1, 2).$$

We say that $V_1$ is an **ideal** in $V_2$ if $V_1 \subset V_2$ and moreover

$$0 \leq u_2 \leq u_1,\ u_1 \in V_1,\ u_2 \in V_2 \quad \text{implies} \quad u_2 \in V_1.$$

**Theorem 11.3.1** (Domination criterion)**.** *The following assertions are equivalent.*

*(i)* $0 \leq e^{-tA_1} \leq e^{-tA_2}$ $(t \geq 0)$.

*(ii)* $V_1$ *is an ideal in* $V_2$ *and* $a_2(u, v) \leq a_1(u, v)$*, whenever* $u, v \in V_1 \cap H_+$.

We refer to the comments for the proof.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We had already shown that

$$0 \leq e^{t\Delta_\Omega^D} \leq e^{t\Delta_\Omega^N} \qquad (t \geq 0), \tag{11.5}$$

see Theorem 4.2.1. Since the form domain of $\Delta_\Omega^D$ is $H_0^1(\Omega)$ and the form domain of $\Delta_\Omega^N$ is $H^1(\Omega)$, we deduce the following from Theorem 11.3.1.

**Proposition 11.3.2.** *The space* $H_0^1(\Omega)$ *is a closed ideal in* $H^1(\Omega)$*.*

Of course, it is easy to give a direct proof of this proposition (which is implicit in Lemma 4.2.3).

With the help of Theorem 11.3.1 we now can extend the domination of the semigroup associated with Neumann by the one associated with Dirichlet boundary conditions from the Laplacian to all elliptic operators with bounded measurable coefficients as considered in Section 11.2.

**Example 11.3.3** (Dirichlet–Neumann domination)**.** *Let* $\mathcal{A}$ *be an elliptic operator defined as in* (11.4) *on* $L^2(\Omega)$*, where* $\Omega$ *is open. Denote by* $A_\Omega^D$ *the realization of* $\mathcal{A}$ *with Dirichlet boundary conditions and by* $A_\Omega^N$ *the realization of* $\mathcal{A}$ *with Neumann boundary conditions. Then*

$$0 \leq e^{-tA_\Omega^D} \leq e^{-tA_\Omega^N} \qquad (t \geq 0). \tag{11.6}$$

*Proof.* The form domain of $A_\Omega^D$ is $H_0^1(\Omega)$ and the form domain of $A_\Omega^N$ is $H^1(\Omega)$. Since $H_0^1(\Omega)$ is an ideal in $H^1(\Omega)$ and since the two forms coincide on $H_0^1(\Omega)$, criterion (ii) of Theorem 11.3.1 is satisfied. ∎

**Example 11.3.4** (monotonicity in the absorption rate). *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two elliptic operators with the same coefficients $a_{ij}$, $b_j$, $c_j$, $i, j = 1, \dots, n$. Assume that the 0-order coefficient $c_0$ of $\mathcal{A}_1$ and the 0-order coefficient $c_0'$ of $\mathcal{A}_2$ satisfy*

$$c_0'(x) \le c_0(x) \qquad a.e.$$

*Let $V$ be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Let $A_1$ be the operator $\mathcal{A}_1$ with form domain $V$, and $A_2$ be the operator $\mathcal{A}_2$ with form domain $V$. Then*

$$0 \le e^{-tA_1} \le e^{-tA_2} \qquad (t \ge 0).$$

This follows directly from Theorem 11.3.1. If $\Omega$ is bounded, connected and has the extension property, then we deduce from Theorem 10.2.10 that $s(A_1) < s(A_2)$ unless $c_0' = c_0$ a.e.

## 11.4 Unbounded drift

In this section we allow the first order coefficients to be unbounded. The technique will consist in compensating by a large 0-order coefficient, which models absorption. This absorption has to be strong enough in order to apply form methods.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $H$ the real space $L^2(\Omega)$. Let $a_{ij} \in L^\infty(\Omega)$ be real coefficients satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n) \tag{11.7}$$

for a.e. $x \in \Omega$. Let $b_j, c_j \in C^1(\Omega)$, $j = 1, \dots, n$, real coefficients, and let $c_0 \in L^1_{\text{loc}}(\Omega)$, $c_0 \ge 0$. We consider the elliptic operator $A_{\min} : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)'$ given by

$$A_{\min}u := -\sum_{i,j=1}^n D_j(a_{ij}D_iu) + \sum_{j=1}^n (b_jD_ju - D_j(c_ju)) + c_0u.$$

We assume that

$$\text{div}b \le c_0 \qquad \text{and} \qquad \text{div}c \le c_0, \tag{11.8}$$

where $\text{div}b = \sum_{j=1}^n D_jb_j$.

Under this condition we show that $A_{\min}$ has an extension $A_p$ such that $-A_p$ generates a positive contraction $C_0$-semigroup $(e^{-tA_p})_{t\ge0}$ on $L^p(\Omega)$, $1 \le p < \infty$. These semigroups are consistent.

These extensions $A_p$ of $A_{\min}$ satisfy Dirichlet boundary conditions in some weak sense. The case where $\Omega = \mathbb{R}^n$ is included and of particular interest.

At first we assume that $c_0$ satisfies the following two hypotheses.

$$|b| \leq \gamma c_0^{\frac{1}{2}}, \ |c| \leq \gamma c_0^{\frac{1}{2}}, \qquad \text{a.e.,} \tag{11.9}$$

where $\gamma > 0$, and

$$\frac{1}{2}(\mathrm{div} b + \mathrm{div} c) \leq \beta c_0, \tag{11.10}$$

where $0 \leq \beta < 1$. Here

$$|b(x)| := \left( \sum_{j=1}^{n} |b_j(x)|^2 \right)^{\frac{1}{2}} \qquad (x \in \Omega).$$

Under these conditions we may define an extension of $A_{\min}$ by a suitable coercive form.

In fact, we define the real space

$$V := \{u \in H_0^1(\Omega) : \int_\Omega c_0|u|^2 < \infty\},$$

which is a Hilbert space for the scalar product

$$(u \mid v)_V := (u \mid v)_{H^1} + \int_\Omega c_0 uv.$$

Then $\mathcal{D}(\Omega) \subset V \overset{d}{\hookrightarrow} L^2(\Omega)$. We need the following.

**Lemma 11.4.1.** *The space $\mathcal{D}(\Omega)$ of all test functions is dense in $V$.*

*Proof.* Let $u \in V_+$. There exists $(v_k)_{k\in\mathbb{N}} \in \mathcal{D}(\Omega)$ such that $v_k \to u$ in $H_0^1(\Omega)$ and a.e. as $k \to \infty$. Let $u_k = (v_k \wedge u) \vee 0$. Then $u_k \in H_{00}^1(\Omega)$, where $H_{00}^1(\Omega)$ denotes the subspace of $H_0^1(\Omega)$ of the functions with compact support, and $u_k \to u$ in $H_0^1(\Omega)$ and a.e. as $k \to \infty$. By the Dominated Convergence Theorem $\int_\Omega c_0|u_k - u|^2 dx \to 0$ as $k \to \infty$, thus $\|u_k - u\|_V \to 0$ as $k \to \infty$. Since $u^+, u^- \in V$ for all $u \in V$, it follows that $H_{00}^1(\Omega)$ is dense in $V$.

Now let $u \in H_{00}^1(\Omega)$. Then $u_k := \rho_k * u \to u \in H_0^1(\Omega)$ as $k \to \infty$. Since $u_k$ vanishes outside a compact subset for $k \geq k_0$, it follows that $\int_\Omega c_0|u_k - u|^2 \to 0$ as $k \to \infty$. Since $u_k \in \mathcal{D}(\Omega)$, the proof is completed. □

Define a bilinear form $a : V \times V \to \mathbb{R}$ by

$$a(u,v) := a_0(u,v) + \int_\Omega \left( \sum_{j=1}^{n} (b_j D_j uv + c_j u D_j v + c_0 uv \right)$$

where

$$a_0(u,v) := \int_\Omega \sum_{j=1}^{n} a_{ij} D_i u D_j v.$$

**Proposition 11.4.2.** *Assume that* (11.9)– (11.10) *hold. Then the form a is continuous, elliptic and accretive.*

*Proof.* a) We show that the form is continuous. Let $u, v \in V$. Then by (11.9)

$$\begin{aligned} \left| \int_\Omega \sum_{j=1}^n b_j D_j uv \right| &\leq \int_\Omega |b||\nabla u||v| \\ &\leq \gamma \int_\Omega c_0^{\frac{1}{2}} |v||\nabla u| \\ &\leq \frac{\gamma}{2} \int_\Omega (c_0|v|^2 + |\nabla u|^2) \\ &\leq \frac{\gamma}{2} (\|v\|_V^2 + \|u\|_V^2). \end{aligned}$$

Replacing $v$ by $\varepsilon v$ and $u$ by $\frac{1}{\varepsilon} u$ we obtain

$$\left| \int_\Omega \sum_{j=1}^n b_j D_j uv \right| \leq \frac{\gamma}{2} \left( \varepsilon^2 \|v\|_V^2 + \frac{1}{\varepsilon^2} \|u\|_V^2 \right).$$

Taking $\varepsilon^2 = \frac{\|u\|_V}{\|v\|_V}$, one obtains

$$\left| \int_\Omega \sum_{j=1}^n b_j D_j uv \right| \leq \gamma \|u\|_V \|v\|_V.$$

The other terms are estimated similarly.

b) The form is elliptic. First we note that

$$a_0(u) \geq \alpha \|\nabla u\|_{L^2(\Omega)}^2 \qquad (u \in V).$$

Using (11.10) we obtain that for $u \in \mathcal{D}(\Omega)$

$$\begin{aligned} a(u) &\geq \alpha \int_\Omega |\nabla u|^2 dx + \int_\Omega \left( \sum_{j=1}^n (b_j + c_j) D_j uu + c_0|u|^2 \right) \\ &= \alpha \int_\Omega |\nabla u|^2 dx + \int_\Omega \left( \sum_{j=1}^n (b_j + c_j) \frac{1}{2} D_j u^2 + c_0|u|^2 \right) \\ &= \alpha \int_\Omega |\nabla u|^2 dx + \int_\Omega \left( -\frac{1}{2} (\mathrm{div} b + \mathrm{div} c) u^2 + c_0|u|^2 \right) \\ &\geq \alpha \int_\Omega |\nabla u|^2 dx + (1 - \beta) \int_\Omega c_0|u|^2 \\ &\geq \min\{\alpha, (1 - \beta)\} \|u\|_V^2 - \alpha \|u\|_{L^2}^2. \end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $V$ it follows that

$$a(u) + \alpha\|u\|_{L^2}^2 \geq \min\{\alpha, (1-\beta)\}\|u\|_V^2$$

for all $u \in V$. Thus $a$ is elliptic.

The above estimate also shows that $a(u) \geq 0$ for $u \in \mathcal{D}(\Omega)$ and hence, by density arguments, for $u \in V$. $\square$

**Remark 11.4.3.** *If $\Omega$ is bounded, then the estimate shows that the form $a$ is actually coercive.*

Define $\mathcal{A} : H^1(\Omega) \to \mathcal{D}(\Omega)'$ by

$$\mathcal{A}u = -\sum_{i,j=1}^{n} D_j(a_{ij}D_i u) + \sum_{j=1}^{n}(b_j D_j u - D_j(c_j u)) + c_0 u. \tag{11.11}$$

Observe that for $u \in H^1(\Omega)$, $c_j u \in L^1_{\mathrm{loc}}(\Omega)$, hence $D_j(c_j u) \in \mathcal{D}(\Omega)'$. So the operator $\mathcal{A}$ is well-defined.

**Theorem 11.4.4.** *Assume that* (11.9)– (11.10) *hold and $\Omega$ is connected. The operator $A$ on $L^2(\Omega)$ associated with $a$ is given by*

$$\begin{cases} D(A) &:= \{u \in V : \mathcal{A}u \in L^2(\Omega)\} \\ Au &:= \mathcal{A}u \end{cases}. \tag{11.12}$$

*The semigroup $(e^{-tA})_{t\geq 0}$ and its adjoint are submarkovian. There exist consistent, contractive, positive, irreducible $C_0$-semigroups $(e^{-tA_p})_{t\geq 0}$ on $L^p(\Omega)$, $1 \leq p < \infty$, such that $A_2 = A$. For all $1 \leq p < \infty$ one has $A_p \subset A$. If $0 < f \in L^p(\Omega)$, then $e^{-tA_p}f \gg 0$ for all $t > 0$, $1 \leq p < \infty$.*

*Proof.* a) By the preceding proposition the form $a$ is continuous and elliptic. Denote by $A$ the operator associated with $a$. Let $u \in D(A)$. Then $u \in V$ and $a(u,v) = (Au \mid v)_{L^2}$ for all $v \in V$. Taking $v \in \mathcal{D}(\Omega)$ we see that $Au = \mathcal{A}u$. Conversely, assume that $u \in V$ such that $\mathcal{A}u \in L^2(\Omega)$. Then by the definition of $\mathcal{A}u$ as distribution $(\mathcal{A}u \mid v) = a(u,v)$ for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $V$ it follows that $u \in D(A)$ and $\mathcal{A}u = Au$. We have shown that the operator $A$ is given by (11.12).

b) Let $u \in V$. Then $u^+, (u-1)^+ \in V$. Now by the proof of Example 9.3.4 one has $a(u \wedge 1, (u-1)^+) \leq 0$ and $a((u-1)^+, u \wedge 1) \leq 0$. This shows that the semigroup $(e^{-tA})_{t\geq 0}$ and $(e^{-tA^*})_{t\geq 0}$ are submarkovian.

c) It follows from Proposition 11.1.2 and Theorem 10.1.5 that the semigroup $(e^{-tA})_{t\geq 0}$ is irreducible. Hence also the extrapolation semigroups are irreducible. In fact, since the semigroup on $L^2(\Omega)$ is holomorphic, for $0 < f \in L^1(\Omega) \cap L^\infty(\Omega)$ one has $e^{-tA_p}f = e^{-tA_2}f \gg 0$ for all $t > 0$, $1 \leq p < \infty$. Since for each $0 < g \in L^p(\Omega)$ there exists $0 < f \leq g$, $f \in L^1(\Omega) \cap L^\infty(\Omega)$, it follows that $e^{-tA_p}g \geq e^{-tA_p}f \gg 0$ for all $t > 0$.

d) Finally, we show that $A_{\min} \subset A_p$. For $p = 2$ this follows from a). Hence, for $u \in \mathcal{D}(\Omega)$, $v = A_{\min}u$, $e^{-tA}u - u = \int_0^t e^{-sA}vds$ for all $t \geq 0$. Since $v \in L^p(\Omega)$ for all $1 \leq p < \infty$, and since the semigroups are consistent, we deduce that

$$e^{-tA_p}u - u = \int_0^t e^{-sA_p}vds \qquad (t \geq 0).$$

By Proposition 2.2.4 this implies that $u \in D(A_p)$ and $A_pu = v$. $\square$

Now we drop the assumptions (11.9)–(11.10) and merely assume

$$\text{div}b \leq c_0, \ \text{div}c \leq c_0 \qquad \text{a.e.} \tag{11.13}$$

Then for $k \in \mathbb{N}$ we consider

$$c_0^k = (1 + \frac{1}{k})c_0 + \frac{1}{k}(|b| + |c|) \in L^1_{\text{loc}}(\Omega),$$

which satisfies (11.9)–(11.10). Then $c_0^k \to c_0$ in $L^1_{\text{loc}}(\Omega)$ as $k \to \infty$. By this we mean that

$$\int_K |c_0^k - c_0|dx \to 0 \qquad \text{as } k \to \infty$$

for all compact subsets $K \subset \Omega$. Moreover, call $a_k$ the form $a$ with $c_0$ replaced by $c_0^k$, i.e., $a_k$ is defined on $V_k \times V_k$ by

$$a_k(u, v) := a(u, v) + \int_\Omega (c_0^k - c_0)uv,$$

where $V_k := \{u \in H_0^1(\Omega) : \int_\Omega |u|^2 c_k dx < \infty\}$. Then $a_k$ is continuous, elliptic, and accretive. Denote by $A_k$ the operator associated with $a_k$ on $L^2(\Omega)$ and by $(e^{-tA_{p,k}})_{t\geq 0}$ the positive extrapolation semigroup on $L^p(\Omega)$. Then for $1 \leq p < \infty$

$$\|e^{-tA_{p,k}}\| \leq 1 \qquad (t \geq 0).$$

Since $V_k$ is clearly an ideal in $V_{k+1}$ and $a_k(u, v) - a_{k+1}(u, v) = \int_\Omega (c_0^k - c_0^{k+1})uv \geq 0$ for all $0 \leq u, v \in V_k$ it follows from Theorem 11.3.1 that

$$0 \leq e^{-tA_{p,k}} \leq e^{-tA_{p,k+1}} \qquad (t \geq 0) \tag{11.14}$$

for $p = 2$. But then (11.14) remains true for all $1 \leq p < \infty$ by consistency. Now we use the following.

**Theorem 11.4.5.** *Let $1 \leq p < \infty$. Let $(T_k(t))_{t\geq 0}$ be contraction $C_0$-semigroups on $L^p(\Omega)$, $k \in \mathbb{N}$, such that*

$$0 \leq T_k(t) \leq T_{k+1}(t) \qquad (t \geq 0).$$

*Then $T(t) = \lim_{k\to\infty} T_k(t)$ exists strongly and defines a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega)$.*

We postpone the proof of Theorem 11.4.5 and conclude in our situation that

$$T_p(t)f := \lim_{k\to\infty} e^{-tA_{p,k}} f$$

exists in $L^p(\Omega)$ for all $f \in L^p(\Omega)$ and defines a $C_0$-semigroup on $L^p(\Omega)$ whose generator we denote by $A_p$. Since $A_{\min} \subset A_{p,k}$ for all $k$ it follows also that $A_{\min} \subset A_p$. In fact, let $u \in \mathcal{D}(\Omega)$, $A_{\min} u = v$. Then

$$e^{-tA_{p,k}}u - u = \int_0^t e^{-sA_{p,k}} v ds \qquad (t \geq 0).$$

Letting $k \to \infty$ we conclude that

$$e^{-tA_p}u - u = \int_0^t e^{-sA_p} v ds \qquad (t \geq 0).$$

By Proposition 2.2.4 this implies that $u \in D(A_p)$ and $A_p u = v$. Finally, it follows from (11.14) that

$$0 \leq e^{-tA_{p,k}} \leq e^{-tA_p} \qquad (t \geq 0)$$

for all $k \in \mathbb{N}$ and $t \geq 0$. Since the semigroup $(e^{-tA_{p,k}})_{t\geq 0}$ is irreducible, also $(e^{-tA_p})_{t\geq 0}$ is irreducible. We have shown the following.

**Theorem 11.4.6.** *Assume* (11.13) *and let* $\Omega$ *be connected. There exists an operator* $A_p \supset A_{\min}$ *which generates a positive, irreducible, contractive* $C_0$*-semigroup on* $L^p(\Omega)$*,* $1 \leq p < \infty$*.*

We conclude this lecture by the proof of Theorem 11.4.5.

*Proof of Theorem 11.4.5.* The strong limit exists by the Beppo Levi Theorem. Then $T(t) \in \mathcal{L}(L^p(\Omega))$ and $T(t+s) = T(t)T(s)$ for $t, s \geq 0$. It remains to prove strong continuity. Let $t_n \downarrow 0$, $0 \leq f \in L^p(\Omega)$. We have to show that $f_n := T(t_n)f \to f$ as $n \to \infty$. Let $g_n := T_1(t_n)f$. Then $0 \leq g_n \leq f_n$ and $g_n \to f$ as $n \to \infty$. Moreover, $\|g_n\|_{L^p} \leq \|f\|_{L^p}$.

a) Let $p = 1$. Then

$$\int_\Omega (f_n - g_n) + \int_\Omega g_n = \int_\Omega f_n \leq \|f\|_{L_1}.$$

Since $\int_\Omega g_n \to \|f\|_{L^1}$, it follows that $\|f_n - g_n\|_{L^1} = \int (f_n - g_n) \to 0$ as $n \to \infty$. Since $g_n \to f$ in $L^1(\Omega)$ also $f_n \to f$ in $L^1(\Omega)$.

b) Let $1 < p < \infty$. It suffices to show that each subsequence of $(f_n)_{n\in\mathbb{N}}$ has a subsequence converging to $f$ in $L^p(\Omega)$. Since $L^p(\Omega)$ is reflexive, we may assume that $(f_n)_{n\in\mathbb{N}}$ converges weakly to a function $h \in L^p(\Omega)$ (consider a subsequence otherwise). Since $g_n \leq f_n$ and $g_n \to f$ it follows that $f \leq h$. Hence $\|f\|_{L^p} \leq \|h\|_{L^p}$. Since $L^p(\Omega)$ is uniformly convex, this implies that $(f_n)_{n\in\mathbb{N}}$ converges strongly to $h$. It follows that $\|h\|_{L^p} \leq \|f\|_{L^p}$. Since $f \leq h$, this implies that $f = h$. $\square$

## 11.5 Exercises

In the first exercise we show that in most cases irreducibility is actually equivalent to the open set $\Omega$ being connected.

**Exercise 11.5.1** (local forms and irreducibility)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $(a, V)$ a continuous, densely defined elliptic form on $H = L^2(\Omega)$. Assume that $a$ is* **local**, *i.e.,*

$$u \in V \quad \text{implies} \quad u^+ \in V \text{ and } a(u^+, u^-) = 0.$$

*Denote by $A$ the associated operator.*

*a) Show that $(e^{-tA})_{t\geq 0}$ is irreducible if and only if $1_\omega V \subset V$ implies $|\omega| = 0$ or $|\Omega \setminus \omega| = 0$ for all Borel measurable sets $\omega$.*

*b) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(a, V)$ be a form on $L^2(\Omega)$ given by* (11.1)*, where $V$ is a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Assume that $u \in V$ implies $u^+ \in V$, so that $e^{-tA_V} \geq 0$ $(t \geq 0)$. Show that the semigroup $(e^{-tA_V})_{t\geq 0}$ is irreducible if $\Omega$ is connected and find an example of a disconnected $\Omega$ such that $(e^{-tA_V})_{t\geq 0}$ is still irreducible.*

In the second and third exercise we study how irreducibility is preserved by domination.

**Exercise 11.5.2** (domination and irreducibility)**.**

1. *Let $(S(t))_{t\geq 0}$, $(T(t))_{t\geq 0}$ be $C_0$-semigroups on $L^p(\Omega)$, $1 \leq p < \infty$, such that*

   $$0 \leq S(t) \leq T(t) \qquad (t \geq 0). \tag{11.15}$$

   *Assume that $(S(t))_{t\geq 0}$, is irreducible. Show that $(T(t))_{t\geq 0}$ is irreducible.*

2. *Give a second proof of a) using the characterisation in Theorem 10.1.5, in the case where $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are associated with elliptic forms.*

3. *Let $(a, V)$ be an elliptic, continuous, densely defined form on $L^2(\Omega)$ such that the associated semigroup $(T(t))_{t\geq 0}$ is positive and irreducible. Let $c_0 : V \to \mathbb{R}_+$ be measurable. Consider*

   $$V_0 := \{u \in L^2(\Omega) : \int_\Omega c_0 |u|^2 dx < \infty\}.$$

   *Assume $V_1 := V_0 \cap V$ to be dense in $L^2(\Omega)$. Consider the form $b(u, v) := a(u, v) + \int_\Omega c_0 uv$ with domain $V_1$. Show that $b$ is continuous and elliptic and denote by $(S(t))_{t\geq 0}$ the associated semigroup. Show that* (11.15) *holds.*

4. *Find an example which satisfies the assumption of 3. such that the semigroup $(S(t))_{t\geq 0}$ is not irreducible.*
   *(Hint: Consider $c_0(x) := x^{-2}$, $\Omega = (-1, 1)$, $V = H_0^1(\Omega)$)*

**Exercise 11.5.3.** *Find two semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ such that* (11.15) *holds, $(T(t))_{t\geq 0}$ is irreducible, but $(S(t))_{t\geq 0}$ is not. (Hint: use Exercise 11.5.1 (a).)*

The last exercise is rather a question concerning an alternative abstract proof of Lemma 11.4.1.

**Question 11.5.4.** *Let $X, Y$ be Banach spaces and $D \subset X \cap Y$ be a subspace such that $D$ is dense in $X$ and also in $Y$. Consider the Banach space $X \cap Y$ with norm $\|x\| = \|x\|_X + \|x\|_Y$. Is $D$ is dense in $X \cap Y$?*

**Answer** (by Jan Maas, Delft):
The following example shows that the answer is no.

Let $X = L^2[-1,1] \cap C[0,1]$ and $Y = L^2[-1,1] \cap C[-1,0]$ be endowed with the norms $\|f\|_X = \|f\|_{L^2[-1,1]} + \|f\|_{L^\infty[0,1]}$ and $\|f\|_Y = \|f\|_{L^2[-1,1]} + \|f\|_{L^\infty[-1,0]}$ respectively. Then $X \cap Y$ equals $C[-1,1]$ endowed with a norm equivalent to the supremum norm.

We define $D = \{f \in C[-1,1] : f(-1) = f(1)\}$. Observe that $D$ is dense in both $X$ and $Y$ but not in $X \cap Y$.

**Exercise 11.5.5** (the dual of intersection)**.**

*a) Show that for each $\phi \in (X \cap Y)'$ there exists $x' \in X'$, $y' \in Y'$ such that $\phi = x' + y'$ (by which we mean $\phi(x) = x'(x) + y'(x)$). In order to do so consider the closed subspace $Z := \{(x, x) : x \in X \cap Y\}$ of $X \oplus Y$. Consider $\phi$ as a functional on $Z$ and extend it by the Hahn–Banach Theorem. Thus the dual space of $X \cap Y$ is $X' + Y'$. However, this seems to be of little help for the question above.*

*b) Assume that the answer of the previous question is yes. Give an alternative short proof of Lemma 11.4.1.*

## 11.6 Comments

The domination criterion of Section 11.3 is due to Ouhabaz. We refer to [Ouh05] for the proof and also to various other domination criteria.

Elliptic operators with unbounded drift were considered in [AMP06], where the same form methods were used. However, the semigroups are approximated by changing the domain $\Omega$ instead of the absorption term $c_0$. This method is used systematically in [Vog01], see also [SB02]. Theorem 11.4.5 is taken from [AGG06], but such arguments were known before, see e.g. [Voi86].

# Lecture 12

# Ultracontractivity

In this lecture we consider ultracontractive semigroups, i.e., semigroups $(T(t))_{t\geq 0}$ on $L^1(\Omega)$ satisfying an estimate of the form

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-n/2} \quad (t>0). \tag{12.1}$$

In view of the Dunford–Pettis criterion, this property implies that the semigroup is given by a bounded kernel. In the first section we show the equivalence of $L^p - L^q$ and $L^1 - L^\infty$ estimates. Then we give a simple criterion to prove ultracontractivity if the semigroup is associated with a form. A more refined criterion is obtained using Nash's Inequality. It allows one to show that in the case where $T$ is contractive in all $L^p$-spaces the constant $c$ in (12.1) only depends on the coerciveness constant. There are four sections.

- Interpolation - extrapolation.
- Ultracontractivity for forms.
- Nash's Inequality.
- Elliptic operators with unbounded drift.

## 12.1 Interpolation - Extrapolation

Let $(\Omega, \sigma, \mu)$ be a $\sigma$-finite measure space. We frequently write $L^p$ for $L^p(\Omega), 1 \leq p \leq \infty$. Recall that

$$L^1(\Omega) \cap L^\infty(\Omega) \subset L^p(\Omega) \subset L^1(\Omega) + L^\infty(\Omega)$$

for all $1 \leq p \leq \infty$. Moreover, $L^1(\Omega) \cap L^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \leq p < \infty$. Thus it is frequently sufficient to give estimates for functions in $L^1(\Omega) \cap L^\infty(\Omega)$. Let

$$B : L^1(\Omega) \cap L^\infty(\Omega) \to L^1(\Omega) + L^\infty(\Omega)$$

be a linear mapping. As before, for $1 \leq p_1, p_2 \leq \infty$, we let

$$\|B\|_{\mathcal{L}(L^{p_1},L^{p_2})} = \sup\{\|Bf\|_{L^{p_2}} : f \in L^1 \cap L^\infty \ , \ \|f\|_{L^{p_1}(\Omega)} \leq 1\},$$

which is defined as an element of $[0, \infty]$. If $\|B\|_{\mathcal{L}(L^{p_1},L^{p_2})} < \infty$ and if $1 \leq p_1 < \infty$, then there exists a unique operator

$$\tilde{B} \in \mathcal{L}(L^{p_1}, L^{p_2}) \text{ such that } \tilde{B}_{|_{L^1 \cap L^\infty}} = B.$$

We call $\tilde{B}$ the **extrapolation operator** of $B$. If $B \in \mathcal{L}(L^p)$ where $1 \leq p < \infty$ and if $\|B\|_{\mathcal{L}(L^{p_1},L^{p_2})} < \infty$ then it follows by density, that

$$\|Bf\|_{L^{p_2}} \leq \|B\|_{\mathcal{L}(L^{p_1},L^{p_2})} \|f\|_{L^{p_1}}$$

for all $f \in L^1 \cap L^\infty$. Moreover, $\tilde{B}$ and $B$ coincide on the space $L^{p_1} \cap L^p$.

Next we formulate the Riesz–Thorin interpolation theorem.

**Theorem 12.1.1** (Riesz–Thorin)**.** *Let $B : L^1 \cap L^\infty \to L^1 + L^\infty$. Let*

$$0 < \theta < 1 \ , \ 1 \leq p_1, p_2 \leq \infty \ , \ 1 \leq q_1, q_2 \leq \infty \ ,$$

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \ , \ \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

*Then*

$$\|B\|_{\mathcal{L}(L^p,L^q)} \leq \|B\|^{\theta}_{\mathcal{L}(L^{p_1},L^{q_1})} \cdot \|B\|^{1-\theta}_{\mathcal{L}(L^{p_2},L^{q_2})}.$$

We also note the following particular case.

**Corollary 12.1.2.** *If $\|B\|_{\mathcal{L}(L^1)} \leq M$ and $\|B\|_{\mathcal{L}(L^\infty)} \leq M$, then $\|B\|_{\mathcal{L}(L^p)} \leq M$ for all $1 \leq p \leq \infty$.*

We are mainly interested in the case where $\|B\|_{\mathcal{L}(L^1,L^\infty)} < \infty$. Then by the Dunford–Pettis criterion, there exists a kernel $K \in L^\infty(\Omega \times \Omega)$ such that

$$Bf(x) = \int K(x,y) f(y) d\mu(y) \quad \text{for a.e. } x \in \Omega$$

for all $f \in L^1 \cap L^\infty$.

We first consider an interpolation result. Let $(T(t))_{t \geq 0}$ be a semigroup which operates on all $L^p$-spaces. A typical case is when $(T(t))_{t \geq 0}$ is a $C_0$-semigroup on $L^p(\Omega)$ such that $(T(t))_{t \geq 0}$ and $(T(t)^*)_{t \geq 0}$ are submarkovian. Then $\|T(t)\|_{\mathcal{L}(L^q)} \leq 1$ for all $t \geq 0, 1 \leq q \leq \infty$.

**Proposition 12.1.3** (interpolation)**.** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on $L^1(\Omega)$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1)} \leq 1 \ , \ \|T(t)\|_{\mathcal{L}(L^\infty)} \leq 1 \tag{12.2}$$

*for all $t \geq 0$. Assume that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-n/2} \quad (t \geq 0) \tag{12.3}$$

*where $c \geq 0$ and $n > 0$ is a real number. Then for $1 \leq p < q \leq \infty$*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \leq c^{\frac{1}{p}-\frac{1}{q}} \ t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \quad (t > 0). \tag{12.4}$$

*Proof.* a) Let $\frac{1}{p} = \frac{\alpha}{1} + \frac{1-\alpha}{\infty}$, i.e., $\alpha = \frac{1}{p}$. Then by the Riesz–Thorin Theorem,

$$\begin{aligned} \|T(t)\|_{\mathcal{L}(L^p,L^\infty)} &\leq \|T(t)\|^{\alpha}_{\mathcal{L}(L^1,L^\infty)} \|T(t)\|^{1-\alpha}_{\mathcal{L}(L^\infty)} \\ &\leq c^{\frac{1}{p}} \ t^{-\frac{n}{2}\frac{1}{p}} \quad (t > 0). \end{aligned}$$

b) Let $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{\infty}$, i.e., $\beta = \frac{p}{q}$. Then by the Riesz–Thorin Theorem and a) we obtain

$$\begin{aligned} \|T(t)\|_{\mathcal{L}(L^p,L^q)} &\leq \|T(t)\|^{\beta}_{\mathcal{L}(L^p)} \|T(t)\|^{1-\beta}_{\mathcal{L}(L^p,L^\infty)} \\ &\leq \left(c^{\frac{1}{p}} t^{-\frac{n}{2}\frac{1}{p}}\right)^{1-\frac{p}{q}} \\ &= c^{\frac{1}{p}-\frac{1}{q}} \ t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}. \end{aligned}$$

This concludes the proof. □

The prototype of the semigroups we consider here is the Gaussian semigroup which satisfies an estimate (12.2)–(12.3) and hence also (12.4). It is surprising that it is possible to go back from the interpolation estimate (12.4) to (12.3). This is a consequence of the semigroup property as we will show next.

We call a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega)$ **completely contractive** if

$$\|T(t)\|_{\mathcal{L}(L^1)} \leq 1 \ \text{ and } \ \|T(t)\|_{\mathcal{L}(L^\infty)} \leq 1$$

for all $t \geq 0$. Then it follows that $\|T(t)\|_{\mathcal{L}(L^q)} \leq 1$ for all $t \geq 0$ and all $1 \leq q \leq \infty$. For example, if $(T(t))_{t\geq 0}$ and $(T(t)^*)_{t\geq 0}$ are submarkovian, then $(T(t))_{t\geq 0}$ is completely contractive.

We will use the following rescaling in the proof: If $A$ generates the $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$, then for $\beta > 0$ the operator $\beta A$ generates the $C_0$-semigroup $(T(\beta t))_{t\geq 0}$, see Exercise 2.6.2.

**Theorem 12.1.4** (extrapolation). *For $1 \le p < q \le \infty$ there exist constants $c_{p,q}$ such that the following holds. Let $(T(t))_{t\ge 0}$ be a completely contractive $C_0$-semigroup on $L^2(\Omega)$. Assume that*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \quad (t>0). \tag{12.5}$$

*Then*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le c_{p,q} t^{-\frac{n}{2}} \quad (t>0).$$

*Proof.* The proof goes in two steps.
$1^{st}$ step: extrapolation to $\mathcal{L}(L^1, L^q)$. Let $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{q}.$$

Then for $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}), \beta = \frac{n}{2}(1 - \frac{1}{q})$ one has $\alpha = \theta\beta$. Let $f \in L^1 \cap L^\infty, \|f\|_{L^1} \le 1$. Note that by hypothesis

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le t^{-\alpha} \quad (t>0). \tag{12.6}$$

Define

$$c_f := \sup_{t>0} t^\beta \|T(t)f\|_{L^q}. \tag{12.7}$$

We want to find a bound for $c_f$ which is independent of $f$. Recall the interpolation inequality

$$\|g\|_{L^p} \le \|g\|_{L^1}^\theta \|g\|_{L^q}^{1-\theta},$$

which is valid for each measurable function $g$. Thus by (12.6)–(12.7)

$$\begin{aligned} \|T(t)f\|_{L^q} &= \|T(t/2)T(t/2)f\|_{L^q} \\ &\le (t/2)^{-\alpha}\|T(t/2)f\|_{L^q} \\ &\le (t/2)^{-\alpha}\|T(t/2)f\|_{L^1}^\theta \|T(t/2)f\|_{L^q}^{1-\theta} \\ &\le (t/2)^{-\alpha}(t/2)^{-\beta(1-\theta)} c_f^{1-\theta} \\ &= (t/2)^{-\beta} c_f^{1-\theta} = t^{-\beta} 2^\beta c_f^{1-\theta}\ , \end{aligned}$$

since $\alpha = \theta\beta$. It follows from the definition of $c_f$ that $c_f \le 2^\beta c_f^{1-\theta}$. Hence $c_f \le 2^{\beta/\theta} := a_{p,q}$. We have shown that

$$\|T(t)\|_{\mathcal{L}(L^1,L^q)} \le a_{p,q} t^{-\frac{n}{2}(1-\frac{1}{q})} \quad (t>0). \tag{12.8}$$

$2^{nd}$ step: extrapolation to $\mathcal{L}(L^1, L^\infty)$. It follows from (12.8) that

$$\|T(t)^*\|_{\mathcal{L}(L^{q'},L^\infty)} \le a_{p,q} t^{-\frac{n}{2}(1-\frac{1}{q})} = a_{p,q} t^{-\beta} \quad (t>0).$$

Write

$$a_{p,q}t^{-\beta} = (a_{p,q}^{-\frac{1}{\beta}}t)^{-\beta}.$$

Hence

$$\|T(a_{p,q}^{1/\beta}t)^*\|_{\mathcal{L}(L^{q'},L^\infty)} \le t^{-\beta} = t^{-\frac{n}{2}\frac{1}{q'}} \quad (t > 0),$$

where $q^{-1} + q'^{-1} = 1$. Applying the first step to this semigroup, we deduce that

$$\|T(a_{p,q}^{1/\beta}t)^*\|_{\mathcal{L}(L^1,L^\infty)} \le a_{q',\infty}t^{-n/2} \quad (t > 0).$$

Taking adjoints, it follows that

$$\|T(a_{p,q}^{1/\beta}t)\|_{\mathcal{L}(L^1,L^\infty)} \le a_{q',\infty}t^{-n/2}$$

hence

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le a_{q',\infty}(a_{p,q}^{-1/\beta}t)^{-n/2} = c_{p,q}t^{-n/2}$$

for all $t > 0$ where $c_{p,q} = a_{q',\infty} \cdot c_{p,q}^{n/2\beta}$. This completes the proof. ☐

If in (12.5) a constant occurs we obtain the following more general result by rescaling.

**Corollary 12.1.5** (extrapolation)**.** *Let $(T(t))_{t\ge0}$ be a completely contractive $C_0$-semigroup on $L^2(\Omega)$. Let $1 \le p < q \le \infty$. Assume that*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \quad (t > 0).$$

*Then there exists a constant $\tilde{c}$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le \tilde{c}t^{-\frac{n}{2p}} \quad (t > 0).$$

*The constant $\tilde{c}$ depends merely on $n, p, q$ and on $c$.*

*Proof.* Let $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$. Then

$$ct^{-\alpha} = (c^{-\frac{1}{\alpha}}t)^{-\alpha}.$$

Apply Theorem 12.1.4 to the semigroup $(S(t))_{t\ge0} = (T(c^{\frac{1}{\alpha}}t))_{t\ge0}$ which satisfies (12.5). ☐

Of particular interest is the case when $p = 2$ and $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$.

**Corollary 12.1.6.** *Let $(T(t))_{t\ge0}$ be a completely contractive $C_0$-semigroup on $L^2(\Omega)$ such that*

$$\|T(t)\|_{\mathcal{L}(L^2,L^q)} \le ct^{-\frac{1}{2}} \quad (t > 0). \tag{12.9}$$

*where $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$. Then*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le c_{2,q}c^nt^{-\frac{n}{2}} \quad (t > 0).$$

*Proof.* Observe that $\frac{n}{2}(\frac{1}{2} - \frac{1}{q}) = \frac{1}{2}$. It follows from (12.9) that

$$\begin{aligned} \|T(c^2t)\|_{\mathcal{L}(L^2,L^q)} &\leq c(c^2t)^{-\frac{1}{2}} = t^{-\frac{1}{2}} \\ &= t^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})} \quad (t > 0). \end{aligned}$$

It follows from Theorem 12.1.4 that

$$\|T(c^2t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_{2,q}t^{-\frac{n}{2}} \quad (t > 0).$$

Hence

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_{2,q}(c^{-2}t)^{-\frac{n}{2}} = c_{2,q}c^n t^{-\frac{n}{2}} \quad (t > 0),$$

the claimed inequality. □

## 12.2 Ultracontractivity for forms

Now we apply the results of the previous section to the case where the $C_0$-semigroup is associated with a form. We consider the real space $H = L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. Let $V$ be a Hilbert space such that $V \hookrightarrow L^2(\Omega)$ is dense. We assume that $u \in V$ implies that $u \wedge 1 \in V$. Furthermore, we assume that $n \geq 2$ and

$$V \subset L^q \qquad \text{where } \frac{1}{q} = \frac{1}{2} - \frac{1}{n}.$$

The following criterion for ultracontractivity has the advantage that the $L^1 - L^\infty$-estimate merely depends on the coerciveness constant of the form.

**Theorem 12.2.1.** *There exists a constant $c_V > 0$ which merely depends on $V$ such that the following holds. Let $a : V \times V \to \mathbb{R}$ be bilinear, continuous such that for some $\mu > 0$*

$$a(u) \geq \mu \|u\|_V^2$$

*and $a(u\wedge 1, (u-1)^+) \geq 0$, $a((u-1)^+, u\wedge 1) \geq 0$ for all $u \in V$. Denote by $T$ the semigroup associated with $a$ on $L^2(\Omega)$. Then*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_V \mu^{-n/2} t^{-n/2} \quad (t > 0).$$

We use the following well-known product rule.

**Lemma 12.2.2.** *Let $u \in C^1((a, b); H)$. Then*

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\,\mathrm{Re}(u(t) \mid \dot{u}(t))_H.$$

*Proof.* One has

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}(\|u(t+h)\|_H^2 - \|u(t)\|_H^2) \\
= &\lim_{h\to 0} \frac{1}{h}\{(u(t+h) \mid u(t+h))_H - (u(t) \mid u(t))_H\} \\
= &\lim_{h\to 0} \frac{1}{h}\{((u(t+h) - u(t) \mid u(t+h))_H + (u(t) \mid u(t+h) - u(t))_H\} \\
= &(\dot u(t) \mid u(t))_H + (u(t) \mid \dot u(t))_H \\
= &\overline{(u(t) \mid \dot u(t))_H} + (u(t) \mid \dot u(t))_H \\
= &2\,\mathrm{Re}(u(t) \mid \dot u(t))_H,
\end{aligned}$$

as we have claimed. □

*Proof of Theorem 12.2.1* a) First we observe that $V \hookrightarrow L^q$, i.e., the injection is continuous. This is a simple consequence of the Closed Graph Theorem. In fact, let $v_k \to v$ in $V$ such that $v_k \to w$ in $L^q$ as $k \to \infty$. We have to show that $w = v$. Since $V \hookrightarrow L^2(\Omega)$, there exists a subsequence $v_{k_\ell}$ converging to $v$ a.e. as $\ell \to \infty$. Hence $w = v$ a.e.
b) Since the injection $V \hookrightarrow L^q$ is continuous, there exists a constant $c \geq 0$ such that

$$\|u\|_{L^q} \leq c\,\|u\|_V \qquad (u \in V).$$

Note that $\|T(t)\|_{\mathcal{L}(L^q)} \leq 1$, hence $\|T(\cdot)f\|_{L^q}$ is decreasing for all $f \in L^q$. Consequently, for $f \in V$ one has

$$\begin{aligned}
t\|T(t)f\|_{L^q}^2 &= \int_0^t \|T(t)f\|_{L^q}^2\,ds \leq \int_0^t \|T(s)f\|_{L^q}^2\,ds \leq c^2 \int_0^t \|T(s)f\|_V^2\,ds \\
&\leq c^2/\mu \int_0^t a(T(s)f)\,ds = -c^2/\mu \int_0^t (AT(s)f \mid T(s)f)_{L^2}\,ds \\
&= -c^2/(2\mu) \int_0^t \frac{d}{ds}\|T(s)f\|_{L^2}^2 ds = c^2/(2\mu)(\|f\|_{L^2}^2 - \|T(t)f\|_{L^2}^2) \\
&\leq c^2/(2\mu)\|f\|_{L^2}^2
\end{aligned}$$

where we used Lemma 12.2.2. We have shown that

$$\|T(t)f\|_q \leq \frac{c}{\sqrt{2\mu}} t^{-1/2}\|f\|_{L^2} \quad (t > 0).$$

It follows from Corollary 12.1.6 that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_{2,q} 2^{-n/2} c^n \mu^{-n/2} t^{-n/2} \quad (t > 0).$$

This completes the proof. □

Finally we give the definition of ultracontractivity in a slightly more general setting.

**Definition 12.2.3.** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on $L^p(\Omega)$ where $1 \leq p < \infty$. We say that $(T(t))_{t\geq 0}$ is* **ultracontractive** *if there exist $q \in (p, \infty]$, $\beta > 0$ and $c > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \leq ct^{-\beta} \quad (0 < t \leq 1). \tag{12.10}$$

Then we may let $0 < n \in \mathbb{R}$ be such that $\beta = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ to be consistent with the preceding. In (12.10) we merely consider $0 < t \leq 1$. But if

$$\sup_{0<t\leq 1} \|T(t)\|_{\mathcal{L}(L^q)} < \infty,$$

then it follows that there exist $\omega \in \mathbb{R}, M > 0$ such that

$$e^{-\omega t}\|T(t)\|_{\mathcal{L}(L^p,L^q)} \leq Mt^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$$

for all $t > 0$.

Now the proof of Theorem 12.1.4 shows the following.

**Proposition 12.2.4.** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on $L^p(\Omega)$ where $1 \leq p < \infty$. Assume that*

$$\sup_{0<t\leq 1} \|T(t)\|_{\mathcal{L}(L^1)} < \infty \quad \textit{and}$$
$$\sup_{0<t\leq 1} \|T(t)\|_{\mathcal{L}(L^\infty)} < \infty.$$

*If $(T(t))_{t\geq 0}$ is ultracontractive, then there exist constants $M > 0$ , $\omega \in \mathbb{R}$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq Me^{\omega t}t^{-n/2} \qquad (t > 0).$$

## 12.3 Ultracontractivity via Nash's inequality

The following inequalities can be proved in $\mathbb{R}^n$ with the help of the Fourier transform (see [Rob91, p. 169]).

**Theorem 12.3.1** (Nash's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$. In the latter case we suppose that $\Omega$ has Lipschitz boundary. Then there exists a constant $c_N$ such that*

$$\|u\|_{L^2}^{2+\frac{4}{n}} \leq c_N\|u\|_V^2\|u\|_{L^1}^{\frac{4}{n}} \qquad (u \in V \cap L^1(\Omega)). \tag{12.11}$$

Under the hypotheses on $\Omega$ (even if $\Omega$ is possibly unbounded) there exists an extension operator

$$E \in \mathcal{L}(H^1(\Omega), H^1(\mathbb{R}^n))$$

such that $\|E\|_{\mathcal{L}(L^1(\Omega),L^1(\mathbb{R}^n))} < \infty$, cf. [Ste70] and [Bie00]. This extension property (which is stronger than the one considered in Proposition 11.1.5) allows one to carry over Nash's inequality from $\mathbb{R}^n$ to $\Omega$.

Now let $V$ be a Hilbert space such that $V \overset{d}{\hookrightarrow} L^2(\Omega)$. We assume that $V$ satisfies **Nash's inequality** in the sense that there exists a constant $c_N > 0$ such that (12.11) holds. For example, this is the case if $V \hookrightarrow H^1_0(\Omega)$ without any further condition on the open set $\Omega$; or if $V \hookrightarrow H^1(\Omega)$ and $\Omega$ has Lipschitz boundary. This follows directly from Theorem 12.3.1.

Let $a : V \times V \to \mathbb{R}$ be a continuous, coercive form. Thus there exists $\alpha > 0$ such that

$$a(u) \geq \alpha \|u\|^2 \qquad (u \in V). \tag{12.12}$$

Denote by $(T(t))_{t\geq 0}$ the semigroup associated with $a$.

**Theorem 12.3.2** (ultracontractivity via Nash's inequality)**.** *Assume in addition to* (12.12) *that the semigroups* $(T(t))_{t\geq 0}$ *and* $(T(t)^*)_{t\geq 0}$ *are submarkovian. Then*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq \left(\frac{nc_N}{2\alpha}\right)^{\frac{n}{2}} t^{-\frac{n}{2}} \qquad (t \geq 0). \tag{12.13}$$

*Proof.* Let $f \in L^1(\Omega) \cap L^2(\Omega)$. Then by (12.13)

$$\begin{aligned} \frac{d}{dt}\|T(t)f\|^2_{L^2} &= 2(AT(t)f \mid T(t)f)_{L^2} = -2a(T(t)f) \\ &\leq -2\alpha\|T(t)f\|^2_V \leq -\frac{2\alpha}{c_n}\frac{\|T(t)f\|^{2+\frac{4}{n}}_{L^2}}{\|T(t)f\|^{\frac{4}{n}}_{L^1}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{d}{dt}(\|T(t)f\|^2_{L^2})^{-\frac{2}{n}} &\geq \left(-\frac{2}{n}\right)(\|T(t)f\|^2_{L^2})^{-\frac{2}{n}-1}\left(-\frac{2\alpha}{c_N}\right)\frac{(\|T(t)f\|^2_{L^2})^{1+\frac{2}{n}}}{\|T(t)f\|^{\frac{4}{n}}_{L^1}} \\ &= \frac{4\alpha}{nc_N}\|T(t)f\|^{-\frac{4}{n}}_{L^1} \\ &\geq \frac{4\alpha}{nc_N}\|f\|^{-\frac{4}{n}}_{L^1}, \end{aligned}$$

since $(T(t))_{t\geq 0}$ is contractive on $L^1(\Omega)$. Hence,

$$\begin{aligned} \|T(t)f\|^{-\frac{4}{n}}_{L^2} &= \int_0^t \frac{d}{ds}(\|T(s)f\|^2_{L^2})^{-\frac{2}{n}} ds + (\|f\|^2_{L^2})^{-\frac{2}{n}} \\ &\geq \frac{4\alpha}{nc_N} t\|f\|^{-\frac{4}{n}}_{L^1}. \end{aligned}$$

Hence $\|T(t)f\|_{L^2}^{\frac{4}{n}} \leq \frac{nc_N}{4\alpha} t^{-1} \|f\|_{L^1}^{\frac{4}{n}}$. We have shown that

$$\|T(t)\|_{\mathcal{L}(L^1(\Omega),L^2(\Omega))} \leq \left(\frac{nc_N}{4\alpha}\right)^{\frac{n}{4}} t^{-\frac{n}{4}} \qquad (t \geq 0). \tag{12.14}$$

Applying (12.14) to $(T(t)^*)_{t\geq 0}$ we obtain

$$\|T(t)\|_{\mathcal{L}(L^2(\Omega),L^\infty(\Omega))} \leq \left(\frac{nc_N}{4\alpha}\right)^{\frac{n}{4}} t^{-\frac{n}{4}} \qquad (t \geq 0). \tag{12.15}$$

These two inequalities yield the final estimate

$$\begin{aligned} \|T(t)\|_{\mathcal{L}(L^1,L^\infty)} &\leq \|T\left(\frac{t}{2}\right)\|_{\mathcal{L}(L^1,L^2)} \|T\left(\frac{t}{2}\right)\|_{\mathcal{L}(L^2,L^\infty)} \\ &\leq \left(\frac{nc_N}{2\alpha}\right)^{\frac{n}{2}} t^{-\frac{n}{2}} \qquad (t \geq 0). \end{aligned}$$

This concludes the proof. □

## 12.4 Elliptic operators with unbounded drift

Here we reconsider the example of Section 11.4. Let $\Omega \subset \mathbb{R}^n$ be open. Let $a_{ij} \in L^\infty(\Omega)$ be real coefficients satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

for a.e. $x \in \Omega$ and some $\alpha > 0$. Let $b = (b_1, \ldots, b_n) \in C^1(\Omega, \mathbb{R}^n)$, $c = (c_1, \ldots, c_n) \in C^1(\Omega, \mathbb{R}^n)$. Let $c_0 : \Omega \to \mathbb{R}$ be measurable. We assume that there are constants $0 < \beta < 1$ and $\gamma > 0$ such that

$$\operatorname{div} b \leq \beta c_0, \ \operatorname{div} c \leq \beta c_0 \qquad \text{a.e.} \tag{12.16}$$

and

$$|b| \leq \gamma c_0^{\frac{1}{2}}, \ |c| \leq \gamma c_0^{\frac{1}{2}}, \qquad \text{a.e.} \tag{12.17}$$

As in Section 11.4 we consider

$$V := \{u \in H_0^1(\Omega) : \int_\Omega c_0|u|^2 dx < \infty\}$$

and the form $a : V \times V \to \mathbb{R}$ given by

$$a(u,v) := \int_\Omega \left( \sum_{i,j=1}^n a_{ij} D_i u D_j v + \sum_{j=1}^n (b_j D_j u v + c_j u D_j v + c_0 uv \right).$$

We know from Proposition 11.4.2 that the form is continuous and elliptic. Moreover, by the proof of Proposition 11.4.2,

$$a(u) + \alpha\|u\|_{L^2}^2 \geq \alpha_0\|u\|_V^2 \qquad (u \in V),$$

where $\alpha_0 = \min\{\alpha, 1-\beta\} > 0$.

Thus, the form $a+\alpha$ is coercive. Denote by $A$ the operator associated with $a$. Then $A+\alpha$ is associated with $a+\alpha$. Now observe that $V$ satisfies Nash's inequality in the sense of the preceding section. It follows from Theorem 12.3.2 that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq \left(\frac{nc_N}{2\alpha_0}\right)^{\frac{n}{2}} e^{\alpha t} t^{-\frac{n}{2}} \qquad (t \geq 0). \tag{12.18}$$

## 12.5 Exercises

The first exercise is used in Section 12.2 but rather belongs to Section 2.

**Exercise 12.5.1** (all semigroups are contractive up to renorming)**.** *Let $(T(t))_{t\geq 0}$ be a bounded $C_0$-semigroup on a Banach space $X$. Then*

$$\|x\|_1 := \sup_{t\geq 0}\|T(t)x\|$$

*defines an equivalent norm on $X$ such that*

$$\|T(t)x\|_1 \leq \|x\|_1 \qquad (t \geq 0).$$

Next we investigate some properties of ultracontractive semigroups if the underlying space has finite measure. Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on $L^1(\Omega)$ where $(\Omega, \Sigma, \mu)$ is a finite measure space. Assume that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-\frac{n}{2}} \qquad (t > 0),$$

where $c > 0$, $n > 0$.

**Exercise 12.5.2.** *a) Show that $T(t)$ is compact for all $t > 0$.*

*Let $1 < p < \infty$.*

*b) Show $T(t)L^p(\Omega) \subset L^p(\Omega)$ $(t > 0)$.*

*Let $S(t) := T(t)_{|L^p(\Omega)}$. Show that the following properties hold.*

*c) $S(t+s) = S(t)S(s)$ $(s, t > 0)$;*

*d) $S(t)$ is compact for all $t > 0$;*

*e) There exist $M > 0$ and $\epsilon > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(L^p)} \leq Me^{-\epsilon t} \qquad (t \geq 1).$$

*f) If $\sup_{0<t\leq 1}\|T(t)\|_{\mathcal{L}(L^p)} < \infty$, then $(S(t))_{t\geq 0}$ is a $C_0$-semigroup.*

Next we consider operators of purely second order. We recall that

$$H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$$

for arbitrary open sets in $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$$

if $\Omega$ has the extension property in the simple form of Definition 11.1.4.

**Exercise 12.5.3.** *Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 3$, and let $a_{ij} \in L^\infty(\Omega)$ such that*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^n)$$

*for some $\alpha > 0$ and a.e. $x \in \Omega$. Let $V = H_0^1(\Omega)$ or let $V$ be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Suppose in the latter case that $\Omega$ has the extension property and that $u \in V$ implies $1 \wedge u \in V$. Consider the form*

$$a(u,v) := \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j v \qquad (u, v \in V).$$

*Let $(T(t))_{t\geq 0}$ be the semigroup associated with $a$. Show that there exists a constant $c > 0$ and $\omega \geq 0$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c t^{-\frac{n}{2}} e^{\omega t} \qquad (t > 0).$$

Finally we suggest to reconsider Definition 12.2.3.

**Exercise 12.5.4.** *Give a proof of Proposition 12.2.4. (Hint: show first that $\|T(t)\|_{\mathcal{L}(L^p)} \leq M e^{\omega t}$ $(t \geq 0)$ for $p = 1, \infty$. Replace $(T(t))_{t\geq 0}$ by $(e^{-\omega t}T(t))_{t\geq 0}$ and modify the proof of Theorem 12.1.4.)*

## 12.6 Comments

Ultracontractivity has been studied systematically by Coulhon, Varopoulos and Saloff–Coste. We refer to [VSC93], [Sal02], and [Dav89] for the historical references and many further results. The simple proof of the extrapolation Theorem 12.1.4 is due to Coulhon [Cou90]. One may add the following interesting further equivalences to Proposition 12.2.4 to obtain the following more complete characterisation.

**Theorem 12.6.1.** *Let $(T_p(t))_{t\geq 0}$ be conistent $C_0$-semigroups on $L^p(\Omega)$ with generator $-A_p$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1)} \leq M e^{\omega t}, \qquad \|T(t)\|_{\mathcal{L}(L^\infty)} \leq M e^{\omega t} \qquad (t \geq 0).$$

*The following are equivalent.*

*(i) There exist* $1 \le p, q \le \infty$ *and* $c > 0$ *such that*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \qquad (0 < t \le 1);$$

*(ii) For all* $1 \le p, q \le \infty$ *there exists* $c > 0$ *such that*

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \qquad (0 < t \le 1);$$

*(iii) There exist* $1 < p, q < \infty$ *and* $0 < \alpha < \frac{n}{2p}$ *such that* $D((\omega - A_p)^\alpha) \subset L^q(\Omega)$ *where* $q$ *is defined by* $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$*;*

*(iv) For all* $1 < p, q < \infty$ *and all* $0 < \alpha < \frac{n}{2p}$ *one has* $D((\omega - A_p)^\alpha) \subset L^q(\Omega)$ *where* $q$ *is defined by* $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$*;*

*(v)* $V \subset L^{\frac{2n}{n-2}}(\Omega)$*;*

*where* (v) *is equivalent to* (i)–(iv) *if* $n > 2$*.*

We refer to [Are04, § 7.3.2] for a survey and precise references. The constant $n > 0$ appearing in the ultracontractivity estimate

$$\|T(t)\|_{\mathcal{L}(L^p,L^q)} \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \qquad (0 < t \le 1)$$

is the same as the one appearing in the injections in (iii) – (v).

If $A_p = -\Delta$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, then $D(\Delta_p^k) = W^{2k,p}(\mathbb{R}^n)$ and these equivalences correspond to classical Sobolev Embedding Theorems.

# Lecture 13

# Gaussian Estimates

The aim of this lecture is to establish Gaussian estimates for semigroups generated by elliptic operators of second order. This means that the semigroup $T$ is given by a kernel $k(t,\cdot,\cdot)$ which has an upper bound of the form

$$|k(t,x,y)| \le ce^{\omega t}t^{-n/2}e^{-|x-y|^2/bt}$$

$x, y$ -a.e. for all $t > 0$. These estimates are interesting in their own right. Even though the generator is a quite general elliptic operator, the semigroup is close to the Gaussian semigroup. But later we will see that Gaussian estimates also have most valuable consequences for spectral theory and they imply strong regularity property for the parabolic equation.

In this lecture we want to present a technique for proving Gaussian estimates. We show in the first section that the existence of an upper Gaussian bound is actually equivalent to uniform ultracontractivity of certain semigroups $T^\varrho$ which are obtained by perturbation of the given semigroup $T$. In the second section we prove that the semigroup generated by an elliptic operator with unbounded drift has a Gaussian estimate. We will use merely our first very elementary ultracontractivity criterion providing a very elementary proof. However, for this approach we need some regularity of the coefficients and it only works for Dirichlet boundary conditions. The underlying open set may be arbitrary though. There are two sections:

13.1 Gaussian bounds by ultracontractivity.

13.2 Gaussian bounds for elliptic operators with unbounded drift.

## 13.1 Gaussian bounds by ultracontractivity

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $T$ be a $C_0$-semigroup on $L^2(\Omega)$. The following definition is basic for this and the following sections.

**Definition 13.1.1.** *The semigroup $T$ has a* **Gaussian (upper) bound** *if there exists a kernel $k(t,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ satisfying*

$$|k(t,x,y)| \leq ct^{-n/2} e^{-|x-y|^2/bt} e^{\omega t} \tag{13.1}$$

*$x, y$ -a.e. for all $t > 0$ (where $b, c > 0, \omega \in \mathbb{R}$ are constants) such that*

$$(T(t)f)(x) = \int_\Omega k(t,x,y) f(y) dy \tag{13.2}$$

*$x$ -a.e. for all $t > 0, f \in L^2(\Omega)$.*

In the following remark we reformulate the definition in terms of domination by the Gaussian semigroup without mentioning kernels explicitly.

**Remark 13.1.2.** *Identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^n)$ by extending functions by 0 outside of $\Omega$. Then also $T(t)$ may be seen as an operator on $L^2(\mathbb{R}^n)$ by letting $\tilde{T}(t)f = (T(t)(f_{|\Omega}))^\sim$ where for $g : \Omega \to \mathbb{K}$ , $\tilde{g}$ is the extension by 0 of $g$ to $\mathbb{R}^n$. We identify $T$ and $\tilde{T}$. Assume that $T$ is positive. Then $T$ has a Gaussian upper bound if and only if there exists $b, c > 0, \omega \in \mathbb{R}$ such that*

$$T(t) \leq ce^{\omega t} G(bt) \quad (t > 0) \ . \tag{13.3}$$

*This follows immediately from Corollary 4.1.3. In the case where $T$ is not positive, (13.3) has to be replaced by*

$$|T(t)f| \leq ce^{\omega t} G(bt)|f| \quad (t > 0) \tag{13.4}$$

*for all $f \in L^2(\Omega)$.*

The purpose of this section is to show that the existence of a Gaussian bound is equivalent to ultracontractivity of certain perturbed semigroups. We continue to consider arbitrary semigroups even though later on we are mostly interested in positive semigroups. We first define a distance in $\mathbb{R}^n$ which is equivalent to the Euclidean distance.

Let

$$W := \{\psi \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : \|D_i\psi\|_\infty \leq 1 \ , \ \|D_iD_j\psi\|_\infty \leq 1 \ \text{ for all } \ i,j = 1,\ldots,n\} \ .$$

Then

$$\begin{aligned} d(x,y) &:= \sup\{\psi(x) - \psi(y) : \psi \in W\} \\ &= \sup\{|\psi(x) - \psi(y)| : \psi \in W\} \end{aligned} \tag{13.5}$$

defines a metric in $\mathbb{R}^n$ which is actually equivalent to the Euclidean distance.

**Lemma 13.1.3.** *There exist constants $d_1, d_2 > 0$ such that*

$$d_1|x-y| \le d(x,y) \le d_2|x-y| \quad \text{for all } \ x,y \in \mathbb{R}^n \ . \tag{13.6}$$

We refer to [Rob91, p. 200 - 202] for the proof.

Now let $T$ be a $C_0$-semigroup on $L^2(\Omega)$. For $\varrho \in \mathbb{R}, \psi \in W$ define the $C_0$-semigroup $T^\varrho$ on $L^2(\Omega)$ by

$$T^\varrho(t)f = e^{-\varrho\psi} T(t)(e^{\varrho\psi} f) \ . \tag{13.7}$$

We do omit the dependance of $T^\varrho$ on $\psi$.

**Theorem 13.1.4** (Davies' trick)**.** *The following assertions are equivalent:*

*(i) There exist $c > 0, \omega \in \mathbb{R}$ s.t.*

$$\|T^\varrho(t)\|_{\mathcal{L}(L^1,L^\infty)} \le ce^{\omega(1+\varrho^2)t} \cdot t^{-n/2} \quad (t > 0)$$

*for all $\psi \in W$, $\varrho \in \mathbb{R}$;*

*(ii) $T$ has a Gaussian bound.*

For the proof we use the following lemma.

**Lemma 13.1.5.** *Let $F \subset C(\Omega \times \Omega), f_0 \in C(\Omega \times \Omega)$. Assume that $f_0(x) = \inf_{f\in F} f(x)$ for all $x \in \Omega \times \Omega$. Let $h : \Omega \times \Omega \to \mathbb{R}$ be measurable s.t. $h(x) \le f(x)$ a.e. for all $f \in F$. Then $h(x) \le f_0(x)$ a.e.*

*Proof.* Let $K \subset \Omega \times \Omega$ be compact. We show that $h(x) \le f_0(x)$ a.e. on $K$. Since $\Omega \times \Omega$ can be written as the countable union of compact sets the claim follows from this. We may assume that $f_0 \equiv 0$ (replacing $F$ by $F - f_0$ and $h$ by $h - f_0$ otherwise). Let $m \in \mathbb{N}$. For each $x \in K$ there exists $f_{m,x} \in F$ such that $f_{m,x}(x) < m^{-1}$. Let $U_{m,x} = \{y \in K : f_{m,x}(y) < m^{-1}\}$. Since $U_{m,x}$ is open and $K$ compact, there exist $x_1^m, \cdots x_{p_m}^m \in K$ such that $K = \bigcup_{j=1}^{p_m} U_{m,x_j^m}$. Thus $\inf_{j=1,\cdots,p_m} f_{m,x_j}(y) < m^{-1}$ for all $y \in K$. The set $F_0 = \{f_{m,x_j} : m \in \mathbb{N}, j = 1, \cdots, p_m\}$ is countable and $\inf_{f\in F_0} f(y) = 0$ for all $y \in K$. Since $h(x) \le f(x)$ a.e. for all $f \in F_0$, it follows that $h \le 0$ a.e. $\square$

*Proof of Theorem 13.1.4 (i) $\Rightarrow$ (ii)* The Dunford-Pettis criterion implies that there exists a kernel $k(t,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ such that $T$ is given by (13.2). We write $T \sim k$. Then for $\psi \in W, \varrho \in \mathbb{R}$ one has $T^\varrho \sim k^\varrho$ where

$$k^\varrho(t,x,y) = k(t,x,y)e^{\varrho(\psi(y)-\psi(x))} \ .$$

Hence by $(i)$ and the Dunford-Pettis criterion one has

$$|k^{\varrho}(t,x,y)| \leq c e^{\omega(1+\varrho^2)t} t^{-n/2} \quad (t>0)$$

$x,y$ -a.e. Thus

$$|k(t,x,y)| \leq ct^{-n/2} e^{\omega t} e^{\omega \varrho^2 t \pm \varrho(\psi(x)-\psi(y))}$$

$x,y$ -a.e. for all $\psi \in W, \varrho \in \mathbb{R}$. Hence

$$|k(t,x,y)| \leq ct^{-n/2} e^{\omega t} e^{\omega \varrho^2 t} e^{-\varrho|\psi(x)-\psi(y)|}$$

$x,y$ - a.e. for all $\varrho \geq 0, \psi \in W, t>0$. It follows from (13.6) that $d : \Omega \times \Omega \to \mathbb{R}_+$ is continuous. Thus Lemma 13.1.5 implies that

$$|k(t,x,y)| \leq ct^{-n/2} e^{\omega t} e^{\omega \varrho^2 t} e^{-\varrho d(x,y)} \tag{13.8}$$

$x,y$ -a.e. for all $x,y \in \Omega, \varrho \geq 0$. Given $x,y \in \Omega$ the minimum over $\varrho$ at the right hand side of (13.8) is attained for $\varrho = \frac{d(x,y)}{2\omega t}$. By Lemma 13.1.5 again, it follows that

$$|k(t,x,y)| \leq ct^{-n/2} e^{\omega t} e^{-d(x,y)^2/4\omega t}$$

$x,y$ -a.e. for all $t>0$. Letting $b = 4\omega/d_1^2$, it follows from Lemma 13.1.3. that

$$|k(t,x,y)| \leq ct^{-n/2} e^{\omega t} e^{-|x-y|^2/bt}$$

$x,y$ -a.e. for all $t>0$.
$(ii) \Rightarrow (i)$ Let $\varrho \in \mathbb{R}, \psi \in W$. Then

$$\begin{aligned}
\|T^{\varrho}(t)\|_{\mathcal{L}(L^1,L^\infty)} &= \operatorname*{ess\,sup}_{x,y\in\Omega} |k^{\varrho}(t,x,y)| \\
&\leq \operatorname*{ess\,sup}_{x,y\in\Omega} |k(t,x,y)| e^{|\varrho||\psi(x)-\psi(y)|} \\
&\leq \sup_{x,y\in\Omega} ct^{-n/2} e^{\omega t} e^{-|x-y|^2/bt} e^{|\varrho| d_2(x-y)} \\
&\leq ct^{-n/2} e^{\omega t} e^{\omega_1 \varrho^2 t}
\end{aligned}$$

where $\omega_1 = bd_2^2/4$, since

$$\begin{aligned}
-|x-y|^2/bt &+ |\varrho| d_2 |x-y| = -1/bt\{|x-y|^2 - |\varrho| bt d_2 |x-y|\} = \\
&- 1/bt(|x-y| - |\varrho| bt d_2/2)^2 + \frac{1}{bt}\varrho^2 b^2 t^2 d_2^2/4 \leq \varrho^2 bt d_2^2/4 \; .
\end{aligned}$$

$\square$

## 13.2 Gaussian bounds for elliptic operators with unbounded drift

In the preceding section we saw that Gaussian estimates can be obtained by proving ultracontractivity of perturbed semigroups. A major problem is the control of the constants appearing in these perturbations. Here we use the elementary method presented in Section 12.1, 12.2. It is probably the easiest way to prove Gaussian estimates. It is restricted to Dirichlet boundary conditions and requires some regularity of the coefficients but no regularity of $\Omega$. We will consider elliptic operators with unbounded drift as they were introduced in Section 11.4 and 12.4.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. By $C_b^1(\Omega)$ we denote the space of all bounded functions in $C^1(\Omega)$ with bounded partial derivatives. Let $a_{ij} \in C_b^1(\Omega)$ be real coefficients such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad (\xi \in \mathbb{R}^n)$$

a.e. where $\alpha > 0$. Let $b = (b_1, \cdots, b_n), c = (c_1, \cdots, c_n) \in C^1(\Omega, \mathbb{R}^n)$ be possibly unbounded real coefficients. Let $c_0 : \Omega \to \mathbb{R}_+$ be locally integrable and assume that there are constants $\gamma > 0$ and $0 < \beta < 1$ such that

$$|b| \leq \gamma c_0^{1/2} \ , \ |c| \leq \gamma c_0^{1/2} \quad \text{a.e.} \tag{13.9}$$

$$\operatorname{div} b \leq \beta c_0 \ , \ \operatorname{div} c \leq \beta c_0 \quad \text{a.e.} \tag{13.10}$$

As in Section 11.4 we consider the Hilbert space

$$V := \{u \in H_0^1(\Omega) : \int_\Omega |u|^2 c_0 dx < \infty\} \tag{13.11}$$

with scalar product $(u|v)_V := (\nabla u|\nabla v) + \int_\Omega c_0 uv \ dx$ and the continuous elliptic form $a : V \times V \to \mathbb{R}$ given by

$$\begin{aligned} a(u,v) = a_0(u,v) \quad &+ \quad \int_\Omega \{\sum_{j=1}^{n}(b_j D_j uv + c_j u D_j v) + c_0 uv\} dx \\ \text{where} \qquad a_0(u,v) \quad &= \quad \int_\Omega \sum_{i,j=1}^{n} a_{ij} D_i u D_j v dx \ . \end{aligned}$$

Denote by $A$ the operator associated with $a$ and let $T(t) = e^{-tA} \quad (t \geq 0)$.

**Theorem 13.2.1.** *The semigroup $T$ has a Gaussian bound.*

The proof will be based on Theorem 13.1.4. For $\varrho \in \mathbb{R}, \psi \in W$ we let

$$T^\varrho(t) = e^{-\varrho\psi} T(t) e^{\varrho\psi}$$

where $e^{-\varrho\psi}$ is considered as the multiplication operator $f \mapsto e^{-\varrho\psi} \cdot f$ on $L^2(\Omega)$. By Theorem 13.1.4 the semigroup $T$ has a Gaussian bound if and only if there exist constants $c > 0$, $b > 0$, $\omega \in \mathbb{R}$ such that

$$\|e^{-(1+\varrho^2)\omega t} T^\varrho(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c t^{-n/2} \quad (t > 0) \ . \tag{13.12}$$

In order to prove such an estimate we will use the ultracontractivity criterion Theorem 12.2.1.

Given $\varrho \in \mathbb{R}, \psi \in W$, the semigroup $T^\varrho$ is generated by the operator

$$A^\varrho = e^{-\varrho\psi} A e^{\varrho\psi}$$

with domain $D(A^\varrho) = \{u \in L^2(\Omega) : e^{\varrho\psi} u \in D(A)\}$. Observe that

$$e^{\varrho\psi} V = V \ . \tag{13.13}$$

**Lemma 13.2.2.** *The operator $A^\varrho$ is associated with the form $a^\varrho : V \times V \to \mathbb{R}$ defined by*

$$a^\varrho(u,v) = a_0(u,v) + \int_\Omega \{\sum_{j=1}^n (b_j^\varrho v D_j u + c_j^\varrho u D_j v) + c^\varrho uv\} dx \tag{13.14}$$

*where*

$$\begin{aligned}
b_j^\varrho &= b_j - \varrho \sum_{k=1}^n a_{jk} \psi_k, \\
c_j^\varrho &= c_j + \varrho \sum_{k=1}^n a_{kj} \psi_k \ , \ j = 1, \cdots, n, \\
c^\varrho &= c_0 - \varrho^2 \sum_{i,j=1}^n a_{ij} \psi_i \psi_j + \varrho \sum_{i=1}^n b_i \psi_i - \varrho \sum_{i=1}^n c_i \psi_i.
\end{aligned}$$

*and where $\psi_i = D_i \psi$. This means that for $u, f \in L^2(\Omega)$ one has $u \in D(A^\varrho), A^\varrho u = f$ if and only if $u \in V$ and $a^\varrho(u,v) = (f \mid v)_{L^2}$ for all $v \in V$.*

*Proof.* By definition of $A^\varrho$ one has for $u, f \in L^2(\Omega)$,

$$u \in D(A^\varrho) \ , \ A^\varrho u = f \text{ if and only if } u \in V \text{ and } a(e^{\varrho\psi} u, v) = (e^{\varrho\psi} f \mid v)_{L^2} \text{ for all } v \in V \ .$$

Replacing $v$ by $e^{-\varrho\psi}v$ this is equivalent to $u \in V$ and $a^{\varrho}(u,v) := a(e^{\varrho\psi}u, e^{-\varrho\psi}v) = (f \mid v)_{L^2}$ for all $v \in V$. Thus we have to compute $a(e^{\varrho\psi}u, e^{-\varrho\psi}v)$. Observe that

$$D_j(e^{\varrho\psi}u) = e^{\varrho\psi}(\varrho\psi_j u + D_j u).$$

Hence

$$\begin{aligned} D_i(e^{\varrho\psi}u)D_j(e^{-\varrho\psi}v) &= (\varrho\psi_i u + D_i u)(-\varrho\psi_j v + D_j v) \\ D_j(e^{\varrho\psi}u)e^{-\varrho\psi}v &= (\varrho\psi_j u + D_j u)v \\ e^{\varrho\psi}uD_j(e^{-\varrho\psi}v) &= u(-\varrho\psi_j v + D_j v). \end{aligned}$$

Thus

$$a_0(e^{\varrho\psi}u, e^{-\varrho\psi}v) = a_0(u,v) + \int_\Omega \sum_{i,j=1}^n a_{ij}(-\varrho\psi_j v D_i u + \varrho\psi_i u D_j v - \varrho^2\psi_i\psi_j uv),$$

and

$$\sum_{j=1}^n \{b_j e^{-\varrho\psi} v D_j(e^{\varrho\psi}u) + c_j e^{\varrho\psi} u D_j(e^{-\varrho\psi}v)\} = \sum_{j=1}^n \{b_j v D_j u + b_j v \varrho\psi_j u + c_j u D_j v - c_j \varrho u \psi_j v\}\ .$$

This shows that $a^{\varrho}$ has the form (13.14). □

**Lemma 13.2.3.** *There exist $\mu > 0, \omega \in \mathbb{R}$ such that the form $b^{\varrho}$ given by*

$$b^{\varrho}(u,v) = a^{\varrho}(u,v) + \omega(1+\varrho^2)(u \mid v)_{L^2}$$

*satisfies*

$$b^{\varrho}(u) \geq \mu\|u\|_V^2\ , \tag{13.15}$$

$$b^{\varrho}(u \wedge 1\ ,\ (u-1)^+) \geq 0\ , \tag{13.16}$$

$$b^{\varrho}((u-1)^+\ ,\ u \wedge 1) \geq 0 \tag{13.17}$$

*for all $u \in V$.*

*Proof.* a) Let $u \in V$. Then $a_0(u) \geq \alpha \int_\Omega |\nabla u|^2 dx$ and since $u \in H_0^1(\Omega)$,

$$
\begin{aligned}
\int_\Omega \sum_{j=1}^n (b_j^\varrho u D_j u + c_j^\varrho u D_j u) &= \int_\Omega (\sum_{j=1}^n (b_j^\varrho + c_j^\varrho) \frac{1}{2} D_j u^2) \\
&= -\frac{1}{2} \int_\Omega \sum_{j=1}^n D_j (b_j^\varrho + c_j^\varrho) u^2 \\
&= -\frac{1}{2} \int_\Omega \sum_{j=1}^n D_j (b_j + c_j) u^2 + \frac{\varrho}{2} \int_\Omega \sum_{j=1}^n D_j (\sum_{k=1}^n (a_{jk} - a_{kj}) \psi_k) u^2 \\
&\geq -\beta \int_\Omega c_0 u^2 - \varrho \omega_1 \|u\|_2^2 \\
&\geq -\beta \int_\Omega c_0 u^2 - (1 + \varrho^2) \omega_1 \|u\|_2^2
\end{aligned}
$$

for some $\omega_1 \geq 0$ independent of $\varrho$ and $u$ since $a_{kj}, \psi_k \in C_b^1(\Omega)$. Here we also use that $\operatorname{div} b, \operatorname{div} c \leq \beta c_0$. The 0-order term in $a^\varrho$ is estimated using the hypothesis, $|b|, |c| \leq \gamma c_0^{1/2}$, the fact that $\|\psi_i\|_\infty \leq 1$ and Cauchy's inequality

$$
a \cdot b = \sqrt{\varepsilon} a \cdot \frac{b}{\sqrt{\varepsilon}} \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 ,
$$

which give

$$
\varrho \int_\Omega \sum_{i=1}^n (b_i - c_i) \psi_i |u|^2 \geq -|\varrho| 2\gamma \int_\Omega c_0^{1/2} |u|^2 \geq -\varepsilon \int_\Omega c_0 |u|^2 - \frac{1}{\varepsilon} \varrho^2 4\gamma^2 \int_\Omega |u|^2 .
$$

Putting all together we find for $\varepsilon = \frac{1-\beta}{2}$,

$$
\begin{aligned}
a^\varrho(u) &\geq \alpha \int_\Omega |\nabla u|^2 - \beta \int_\Omega c_0 u^2 - (1 + \varrho^2) \omega_1 \|u\|_2^2 + \int |u|^2 c_0 \\
&- \varrho^2 \int \sum_{i,j=1}^n |a_{ij}| |\psi_i| |\psi_j| |u|^2 - \varepsilon \int_\Omega c_0 |u|^2 - \frac{1}{\varepsilon} \varrho^2 4\gamma^2 \int |u|^2 \\
&\geq \alpha \int_\Omega |\nabla u|^2 + \frac{1-\beta}{2} \int_\Omega c_0 |u|^2 - (1 + \varrho^2) \omega_2 \|u\|_{L^2}^2 \\
&\geq \mu \|u\|_V^2 - (1 + \varrho^2) \omega_2 \|u\|_{L^2}^2
\end{aligned}
$$

for all $u \in V, \varrho \in \mathbb{R}$ for some constant $\omega_2$, where $\mu = \min\{\alpha, \frac{1-\beta}{2}\} > 0$. This finishes the proof of (13.15).

b) We prove (13.16) replacing $\omega_2$ by a larger constant $\omega$. Let $u \in V$. Since

$$D_j(u \wedge 1) = D_j u 1_{\{u<1\}} \ , \ D_j(u-1)^+ = D_j u 1_{\{u>1\}}$$

one has $D_i(u \wedge 1) D_j(u-1)^+ = 0$ and $D_i(u \wedge 1)(u-1)^+ = 0$ a.e. Hence

$$\begin{aligned} a^\varrho\left((u \wedge 1), (u-1)^+\right) = & \int_\Omega \sum_{j=1}^n c_j^\varrho (u \wedge 1) D_j (u-1)^+ \ + \ \int_\Omega c_0^\varrho (u \wedge 1)(u-1)^+ = \\ & \int_\Omega \sum_{j=1}^n c_j^\varrho D_j[(u \wedge 1)(u-1)^+] \ + \ \int_\Omega c_0^\varrho (u \wedge 1)(u-1)^+ = \\ & -\int_\Omega \sum_{j=1}^n D_j c_j^\varrho (u \wedge 1)(u-1)^+ \ + \ \int_\Omega c_0^\varrho (u \wedge 1)(u-1)^+ \ . \end{aligned}$$

Thus it suffices to show that

$$-\sum_{j=1}^n D_j c_j^\varrho + c_0^\varrho \geq -(1+\varrho^2)\omega_5 \quad \text{a.e.}$$

for all $\varrho \in \mathbb{R}$ and some $\omega_5 \geq 0$. Since div $c \leq \beta c_0$ and $|b|, |c| \leq \gamma c_0^{1/2}$ we have

$$\begin{aligned} c_0^\varrho - \sum_{j=1}^n D_j c_j^\varrho &= c_0 - \varrho^2 \sum_{i,j=1}^n a_{ij}\psi_i\psi_j + \varrho \sum_{j=1}^n (b_j - c_j)\psi_j - \operatorname{div} c - \varrho \sum_{j=1}^n D_j \sum_{i=1}^n a_{ij}\psi_i \\ &\geq \ c_0 - \varrho^2\omega_3 - \varrho\sum_{j=1}^n (b_j - c_j)\psi_j - \beta c_0 - (1+\varrho^2)\omega_4 \\ &\geq \ (1-\beta)c_0 - (1+\varrho^2)(\omega_3+\omega_4) - 2\varrho c_0^{1/2} 2\gamma \\ &\geq \ (1-\beta)c_0 - (1+\varrho^2)(\omega_3+\omega_4) - \varepsilon c_0 - \frac{1}{\varepsilon}\varrho^2 \cdot 4\gamma^2 \\ &\geq \ -(1+\varrho^2)\omega_5 \end{aligned}$$

for all $\varrho \in \mathbb{R}$ if $\omega_5 = \omega_3 + \omega_4 + \frac{4}{\varepsilon}\gamma^2$ , $\varepsilon = (1-\beta)$. This proves the estimate (13.16). The estimate (13.17) follows in the same way since the conditions in $b$ and $c$ are symmetric. $\square$

Now we can apply Theorem 12.2.1 to the form $b^\varrho$ and we conclude that

$$\|e^{-\omega(1+\varrho^2)t} T^\rho(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq c_V \mu^{-n/2} t^{-n/2} \quad (t > 0)$$

for all $\psi \in W, \varrho \in \mathbb{R}$. Theorem 13.1.4 implies that $T$ has a Gaussian upper bound. Thus Theorem 13.2.1 is proved.

## 13.3 Exercises

In the first exercise we show that a Gaussian estimate on the interval $[0,1]$ implies Gaussian estimates on $\mathbb{R}_+$.

**Exercise 13.3.1** (exponential bound)**.** *Let $T$ be a $C_0$-semigroup on $L^2(\Omega)$ such that*

$$|T(t)f| \le cG(bt)|f| \quad (0 < t \le 1)$$

*for all $f \in L^2(\Omega)$ where $b, c > 0$. Show that there exist $M \ge 0, \omega \ge 0$ such that*

$$|T(t)f| \le Me^{\omega t}G(bt)|f|$$

*for all $f \in L^2(\Omega)$ and all $t \ge 0$. Here we use the convention of Remark 13.1.2.*

**Hint:** *Imitate the proof of (2.14).*

In the next two exercises one can compute the kernels explicitly and prove Gaussian estimates.

**Exercise 13.3.2** (elliptic operators with constant coefficients)**.** *Let $C = (c_{ij})_{i,j=1\cdots n}$ be a strictly positive definite matrix and define the operator $A$ on $L^2(\mathbb{R}^n)$ by*

$$A = -\sum_{i,j=1}^{n} c_{ij} D_i D_j \ ,$$

*$D(A) = H^2(\mathbb{R}^n)$. Denote by $v_1, \cdots v_n > 0$ the eigenvalues of $C$.*
*a) Show that the kernel of $e^{-tA}$ is given by*

$$k(t,x,y) = (4\pi t)^{-n/2}(v_1 \cdots v_n)^{-\frac{1}{2}} \exp\left(-\left\|C^{-1}(x-y)\right\|^2/4t\right) \ .$$

*b) Deduce from a) that $(e^{-tA})_{t\ge 0}$ has an upper Gaussian estimate.*

**Exercise 13.3.3.** *Let $Af = -f'' + f'$ on the real space $L^2(\mathbb{R})$, i.e., $A$ is associated with the form*

$$a(u,v) = \int_{\mathbb{R}} u'v' + \int_{\mathbb{R}} u'v dx \ , \ V = H^1(\mathbb{R}) \ .$$

*Compute the kernel of $e^{-tA}$ and show that it has an upper Gaussian estimate.*

**Hint:** *Show that $e^{-tA} = G(t)S(t)$ where $(S(t)f)(x) = f(x-t)$.*

The semigroup generated by the Neumann Laplacian does not always have a bounded kernel. Some regularity of $\Omega$ is needed (see Theorem 13.4.1 in the comments). Here is a counterexample.

**Exercise 13.3.4** (the Neumann Laplacian without kernel)**.** *a) Let $A$ be an operator with compact resolvent. Show that* $\dim\ker A < \infty$.
*Let $\Omega = (0,1) \setminus \{\frac{1}{m} : m \in \mathbb{N}\} \subset \mathbb{R}$.*
*b) Show that the Neumann Laplacian $\Delta_\Omega^N$ on $L^2(\Omega)$ does not have compact resolvent.*
*c) Show that for $t > 0$ there is no kernel $k(t,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ such that*

$$(e^{t\Delta_\Omega^N} f)(x) = \int_\Omega k(t,x,y)f(y)dy \quad a.e.$$

$(f \in L^2(\Omega))$.

The proof of Theorem 13.2.1 was based on the elementary characterisation of ultracontractivity given in Section 1 and 2. Here is an alternative way.

**Exercise 13.3.5** (Gaussian estimates via Nash's inequality). *Give a proof of Theorem 13.2.1 via Theorem 12.3.2.*

**Hint:** *Use Theorem 13.1.4, Lemma 13.2.3 and Lemma 13.2.2.*

# 13.4 Comments

Upper and lower Gaussian bounds for non-symmetric elliptic operators with bounded measurable coefficients on $\mathbb{R}^n$ were first proved by Aronson [Aro68] who used Moser's Harnack inequality for the proof. New impetus to the subject came from Davies [Dav87] who introduced the perturbation method Theorem 13.1.4 and proved Gaussian upper bounds with optimal constants for symmetric purely second order operators with $L^\infty$-coefficients for Dirichlet and Neumann boundary conditions (see also [Dav89]). Gaussian bounds for non-symmetric elliptic operators with Dirichlet, Neumann and Robin boundary conditions were proved in [AtE97] by two different approaches to prove uniform ultracontractivity as needed to apply Davies' Theorem 13.1.4. The first is very elementary and is based on the Beurling-Deny criterion but needs some regularity on the coefficients and is restricted to Dirichlet boundary conditions. This method was extended in [AMP06] to unbounded first order coefficients. The proof we give here differs slightly. We use the elementary ultracontractivity estimate of Section 12.1., 12.2 instead of Nash's inequality.

Gaussian estimates hold also for more general boundary conditions. Here is a quite general result. By $C_b^2(\mathbb{R}^n)$ we denote all bounded $C^2$ functions on $\mathbb{R}^n$ with bounded first and second order derivatives.

**Theorem 13.4.1.** *Let $\Omega \subset \mathbb{R}^n$ open, $H = L^2(\Omega)$, and let $V$ be a closed subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Assume that $u \wedge 1 \in V$ for all $u \in V$ and that $\Omega$ has Lipschitz boundary if $V \neq H_0^1(\Omega)$. Assume furthermore that $C_b^2(\mathbb{R}^n)V \subset V$. Let $a_{ij}, b_j, c_j, c_0 \in L^\infty(\Omega), i, j = 1, \ldots, n$ be real coefficients such that*

$$\sum a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad (\xi \in \mathbb{R}^n)$$

*a.e. where $\alpha > 0$. Consider the elliptic continuous form $a : V \times V \to \mathbb{R}$ given by*

$$a(u, v) = \int \{\sum_{i,j=1}^{n} a_{ij} D_i u D_j v + \sum_{j=1}^{n} b_j v D_j u + c_j u D_j v + c_0 uv\} dx \ .$$

*Denote by $A$ the operator associated with $a$. Then $(e^{-tA})_{t\geq 0}$ has an upper Gaussian bound.*

For $b_j, c_j \in W^{1,\infty}(\Omega)$ (also possibly complex-valued) Theorem 13.4.1 was proved by an iteration method in [AtE97] (due to Fabes-Stroock [FS86]) and then by Daners [Dan00] in the general case. Ouhabaz [Ouh04] gave a new proof and generalizations, which are also presented in his book [Ouh05].

# Lecture 14

# Heat Semigroups on $L^1(\Omega)$

The natural space to describe diffusion is $L^1(\Omega)$. The semigroup $T$ generated by an elliptic operator with real coefficients is positive as we saw. Given a positive initial value $f \in L^1(\Omega)$, i.e., an initial density of the substance which diffuses in the liquid, the function $T(t)f$ gives the density at time $t > 0$ and, for $S \subset \Omega$

$$\int_S (T(t)f)(x)dx \tag{14.1}$$

is the amount of the substance in the region $S$. Similarly, if the model is heat conduction, then $f$ is the initial heat distribution and (14.1) is the heat amount in the set $S$. Thus the $L^1$-norm has a physical meaning. For this reason it is important to study diffusion semigroups (which is the same as heat semigroups) on $L^1(\Omega)$. However, for many questions the space $L^1(\Omega)$ turns out to be more difficult then the reflexive $L^p$-spaces. One such delicate property is holomorphy. Let $T_p$ be a consistent family of $C_0$-semigroups on $L^p(\Omega), 1 \leq p < \infty$. Frequently one first constructs $T_2$ (for example by form methods) and $T_p$ is obtained by extrapolation. If $T_2$ is holomorphic, then we have seen in Lecture 4 that also $T_p$ is holomorphic, for $1 < p < \infty$ (see also the comment 4.6.6). However, $T_1$ may fail to be holomorphic, in general. Things are different, if $T_2$ allows a Gaussian estimate. Then holomorphy extrapolates to $L^1$. This is the main result of this Lecture. There are three sections.

14.1 Extrapolation.

14.2 Holomorphy in $L^1$.

14.3 Convergence to an equilibrium.

## 14.1 Extrapolation

The Gaussian semigroup operates on all spaces $L^p(\mathbb{R}^n), 1 \leq p < \infty$ as a $C_0$-semigroup. The same is true for semigroups which have a Gaussian upper bound. Let $\Omega \subset \mathbb{R}^n$ be an open set. We identify $L^p(\Omega)$ with a subspace of $L^p(\mathbb{R}^n)$ extending functions in $L^p(\Omega)$ by 0 outside of $\Omega$.

**Proposition 14.1.1.** *Let $S$ be a $C_0$-semigroup on $L^1(\mathbb{R}^n)$ and let $T$ be a $C_0$-semigroup on $L^2(\Omega)$ such that*

$$|T(t)f| \leq cS(t)|f| \quad (0 < t \leq 1)$$

*for all $f \in L^1(\Omega) \cap L^2(\Omega)$, where $c > 0$. Then there exists a $C_0$-semigroup $T_1$ on $L^1(\Omega)$ which is consistent with $T$.*

*Proof.* It suffices to show that $T(t)f \to f$ in $L^1(\Omega)$ as $t \downarrow 0$ for all $f \in L^1(\Omega) \cap L^2(\Omega)$. Let $f \in L^1(\Omega) \cap L^2(\Omega)$. Let $t_n \to 0$. It suffices to show that $T(t_{n_k})f \to f$ in $L^1(\Omega)$ as $k \to \infty$ for some subsequence. Recall that each convergent sequence in $L^1$ has a dominated subsequence. Since $S(t)|f| \to |f|$ as $t \downarrow 0$ in $L^1(\mathbb{R}^n)$, we may assume that $S(t_n)|f| \leq h$ and some $h \in L^1(\mathbb{R}^n)$ for all $n \in \mathbb{N}$ (taking a subsequence otherwise). Hence $|T(t_n)f| \leq cS(t_n)|f| \leq ch$ for all $n \in \mathbb{N}$. Since $T(t_n)f \to f$ in $L^2(\Omega)$ as $n \to \infty$, there exists a subsequence such that $T(t_{n_k})f \to f$ a.e. as $k \to \infty$. Hence $T(t_{n_k})f \to f$ in $L^1(\Omega)$ by the Dominated Convergence Theorem. □

**Corollary 14.1.2.** *Let $T$ be a $C_0$-semigroup on $L^2(\Omega)$ admitting an upper Gaussian bound. Then there exist $C_0$-semigroups $T_p$ on $L^p(\Omega), 1 \leq p < \infty$, and a dual semigroup $T_\infty$ on $L^\infty(\Omega)$ such that*

$$T_p(t)f = T_q(t)f \text{ for all } t > 0, f \in L^p \cap L^q, 1 \leq p, q \leq \infty \text{ and } T_2(t) = T(t) \quad (t > 0)\ .$$

*Proof.* By assumption there exist $c > 0, b > 0, \omega \geq 0$ such that

$$|T(t)f| \leq ce^{\omega t}G(bt)|f| \quad (t > 0)$$

for all $f \in L^2(\Omega)$. Now the claim follows from Proposition 14.1.1 for $p = 1$. For $1 < p < \infty$, part 1 of the proof of Theorem 4.4.1 works also here. Since also $T^*$ has a Gaussian upper bound, we may define

$$T_\infty(t) = ((T^*)_1(t))'\ .$$

See also the discussion following Theorem 4.4.1. □

We will now use the Gaussian estimates to show that various properties extend from the semigroup $T$ to the semigroups $T_p$ and in particular to $T_1$. Our first point concerns holomorphy.

## 14.2 Holomorphy in $L^1$

Let $\Omega \subset \mathbb{R}^n$ be open. We consider the complex $L^p$ spaces. The purpose of this section is to prove the following.

**Theorem 14.2.1.** *Let $T$ be a holomorphic $C_0$-semigroup on $L^2(\Omega)$ admitting an upper Gaussian bound. Each extrapolation semigroup $T_p$ is a holomorphic $C_0$-semigroup on $L^p(\Omega)$ , $1 \le p < \infty$.*

We will see in the proof that the maximal holomorphy angle (i.e. the maximal angle $\theta_p \in (0, \pi/2]$ such that $T_p$ has a holomorphic locally bounded holomorphic extension to $\Sigma(\theta_p)$ is independent of $p \in [1, \infty)$). For the proof we need several auxiliary results. At first we show that for a holomorphic function of kernel operators one may choose a kernel representation which is a holomorphic function.

**Lemma 14.2.2.** *Let $D \subset \mathbb{C}$ be open and $F : D \to L^\infty(S)$ holomorphic where $(S, \Sigma, \mu)$ is a measure space. Then there exists $f : D \times S \to \mathbb{C}$ such that*

$$\begin{array}{rcl} f(z,\cdot) & \in & L^\infty(S) \quad \textit{for all} \quad z \in D \ , \\ f(z,\cdot) & = & F(z) \quad \textit{a.e. for all} \quad z \in D \ , \\ f(\cdot,x) & : & D \to \mathbb{C} \quad \textit{is holomorphic for all} \quad x \in \Omega \ . \end{array}$$

*Proof.* Let $B = B(z_0, r) = \{z \in \mathbb{C} \ , \ |z - z_\circ| < r\}$ such that $\bar{B} \subset D$. Then there exist $a_n \in L^\infty(S)$ such that $\sum_{n=0}^{\infty} \|a_n\|_\infty r^n < \infty$ and $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Define $h : B \times S \to \mathbb{C}$ by $h(z, x) = \sum_{n=0}^{\infty} a_n(x)(z - z_0)^n$. Then $h(\cdot, x) : B \to \mathbb{C}$ is holomorphic for all $x \in S$ and $h(z, \cdot) = F(z)$ in $L^\infty(S)$. Let $B_1, B_2$ be two such discs such that $\overline{B_1} \subset D$ , $\overline{B_2} \subset D$ and let $h_j : B_j \times D \to \mathbb{C}$ be functions $(j = 1, 2)$ such that $h_j(z, \cdot) = F(z)$ in $L^\infty(S)$ $(z \in B_j)$ and such that $h_j(\cdot, x)$ is holomorphic on $B_j$ for all $x \in S$. If $B_1 \cap B_2 \neq \emptyset$, then $h_1(z, x) = h_2(z, x)$ for all $z \in B_1 \cap B_2$ , $x \in S$ by the identity theorem. Now it suffices to cover $D$ by such discs. ☐

The next criterion is very convenient to prove holomorphy of vector-valued functions. A subset $W$ of $X^*$ is called **separating** if for all $x \in X$, $x \neq 0$ there exists $\varphi \in W$ such that $\varphi(x) \neq 0$.

**Theorem 14.2.3.** *Let $D \subset \mathbb{C}$ be open and $f : D \to X$ be locally bounded such that $\varphi \circ f$ is holomorphic for all $\varphi \in W$ where $W \subset X^*$ is separating. Then $f$ is holomorphic.*

**Proposition 14.2.4.** *Let $D \subset \mathbb{C}$ be open and $T : D \to \mathcal{L}(L^2)$ holomorphic such that*

$$\sup_{z \in K} \|T(z)\|_{\mathcal{L}(L^1, L^\infty)} < \infty$$

*for each compact subset $K$ of $\Omega$. Then there exists a function*

$$k : D \times \Omega \times \Omega \to \mathbb{C} \text{ such that}$$

$$\begin{aligned} k(z,\cdot,\cdot) &\in L^\infty(\Omega\times\Omega) \quad \text{for all} \quad z \in D\ , \\ (T(z)f)(x) &= \int k(z,x,y)f(y)dy \quad \text{a.e. for all} \quad f \in L^1 \cap L^2\ , \\ k(\cdot,x,y) &: D \to \mathbb{C} \quad \text{is holomorphic for all} \quad x,y \in \Omega\ . \end{aligned}$$

*Proof.* There exists a function

$$\tilde{T} : D \to \mathcal{L}(L^1, L^\infty) \text{ such that } \tilde{T}(z)f = T(z)f \text{ for all } f \in L^1 \cap L^2\ ,\ z \in D\ .$$

It follows from the preceding Theorem 14.2.3 that $\tilde{T}$ is holomorphic. Since by the Dunford-Pettis criterion (Theorem 4.1.1), $\mathcal{L}(L^1, L^\infty)$ and $L^\infty(\Omega \times \Omega)$ are isomorphic, we find a holomorphic function $\tilde{k} : D \to L^\infty(\Omega\times\Omega)$ such that $\tilde{T}(z)$ is represented by the kernel $\tilde{k}(z)$ for all $z \in D$. Now the claim follows from Lemma 14.2.2. □

Next we recall the following well-known version of the Phragmen-Lindelöf theorem [Con78, cor. 6.4.4].

**Proposition 14.2.5.** *Let $\gamma \in (0, \frac{\pi}{2})$ and let $D = \{re^{i\theta} : r > 0\ ,\ 0 < \theta < \gamma\}$. Let $k : \bar{D} \to \mathbb{C}$ be continuous, holomorphic on $D$ such that $|h(z)| \le \alpha \exp(\beta|z|) \quad (z \in D)$ where $\alpha, \beta > 0$. If $|h(r)| \le M\ ,\ |h(re^{i\gamma})| \le M$ for all $r > 0$, then $|h(z)| \le M$ for all $z \in D$.*

We need the following consequence.

**Lemma 14.2.6.** *Let $\gamma \in (0, \frac{\pi}{2})$ and let $k : \overline{\Sigma(\gamma)} \to \mathbb{C}$ be continuous, holomorphic in $\Sigma(\gamma) := \{re^\alpha : r > 0\ ,\ |\alpha| < \gamma\}$ such that*

*a) $|k(z)| \le c \quad (z \in \Sigma(\gamma))$ and*

*b) $|k(t)| \le ce^{-b/t} \quad (t > 0)$,*

*where $c, b > 0$. Then for $0 < \theta_1 < \gamma$ one has*

$$|k(z)| \le c \cdot \exp(-b_1/|z|) \quad (z \in \Sigma(\theta_1))$$

*where $b_1 = \frac{\sin(\gamma-\theta_1)}{\sin\gamma} \cdot b$.*

*Proof.* Let $g(z) = k(z^{-1}) \cdot \exp\{be^{i(\frac{\pi}{2}-\gamma)} \cdot z\frac{1}{\sin\gamma}\}$. Then

(i) $|g(r)| \le ce^{-b\cdot r} \exp\{b \operatorname{Re} e^{i(\frac{\pi}{2}-\gamma)} \cdot r\frac{1}{\sin\gamma}\} = c \quad (r > 0)$;

(ii) $|g(re^{i\gamma})| \leq c \cdot \exp\{b \operatorname{Re}(e^{i(\frac{\pi}{2}-\gamma)} re^{i\gamma}) \frac{1}{\sin\gamma}\} = c$,

(iii) $|g(z)| \leq c \exp\{\frac{b}{\sin\gamma}|z|\}$ , $z \in \Sigma(\gamma)$.

It follows from Proposition 4.3 that $|g(z)| \leq c$ for all $z \in D$. Replacing $g$ by $z \mapsto \overline{g(\bar{z})}$, we see that $|g(z)| \leq c$ for all $z \in \Sigma(\gamma)$. Hence for $z = re^{i\theta}$ , $|\theta| \leq \gamma$.

$$\begin{aligned}
|k(z)| &= |g(z^{-1}) \exp(-be^{i(\frac{\pi}{2}-\gamma)} z^{-1} \tfrac{1}{\sin\gamma})| \\
&\leq c \cdot \exp(-b/r \cdot \operatorname{Re} e^{i(\frac{\pi}{2}-\gamma-\theta)} \tfrac{1}{\sin\gamma}) \\
&= c \exp(-b/r \ \sin(\gamma+\theta)/\sin\gamma) \\
&\leq c \exp(-b \cdot \sin(\gamma - |\theta|)/(r \cdot \sin\gamma)) \ .
\end{aligned}$$

□

**Lemma 14.2.7.** *Let $k : \Omega \times \Omega \to \mathbb{C}$ measurable, $h \in L^1(\mathbb{R}^n)$ such that $|k(x,y)| \leq h(x - y)$ $(x, y \in \Omega)$. Then*

$$(B_k f)(x) = \int k(x,y) f(y) dy$$

*defines a bounded operator $B_k$ on $L^p(\Omega)$ and $\|B_k\|_{\mathcal{L}(L^p)} \leq \|h\|_{L^1(\mathbb{R}^n)}$.*

This is immediate from Young's inequality. For the proof of Theorem 14.2.1 we use the following terminology. Let $S = (S(t))_{t \geq 0}$ be a $C_0$-semigroup on a Banach space $X$. We say that $S$ **is holomorphic of angle** $\theta \in (0, \frac{\pi}{2}]$ if $S$ has a holomorphic extension (still denoted by $S$) from the sector $\Sigma(\theta)$ into $\mathcal{L}(X)$ which is locally bounded, i.e., $S$ is bounded on the sets $\{re^{i\alpha} : 0 < r \leq 1 \ , \ |\alpha| \leq \theta - \varepsilon\}$ for all $\varepsilon > 0$. Then the following is easy to show (see Exercise 14.3.1).

$$S(z_1)S(z_2) = S(z_1 + z_2) \quad (z_1, z_2 \in \Sigma(\theta)) \ ; \tag{14.2}$$

$$\lim_{\substack{z \to 0 \\ z \in \Sigma(\theta')}} S(z)x = x \quad (x \in X) \text{ for all } \theta' \in (0, \theta) \tag{14.3}$$

$$\text{for all } \theta' < \theta \text{ there exist } M \geq 0, w \in \mathbb{R} \text{ such that } \|S(z)\| \leq Me^{|z|\omega} \quad (z \in \Sigma(\theta')) \ . \tag{14.4}$$

*Proof of Theorem 14.2.1* Assume that $T$ is a holomorphic $C_0$-semigroup of angle $\theta \in (0, \pi/2]$ on $L^2(\Omega)$ which satisfies a Gaussian estimate. Let $0 < \theta_1 < \theta$. Choose $\theta_1 < \gamma < \theta_2 < \theta_3 < \theta$. Replacing $T$ by $(e^{-wt}T(t))_{t \geq 0}$ we can assume that

$$\|T(z)\|_{\mathcal{L}(L^2)} \leq \text{const} \quad (z \in \Sigma(\theta_3)) \ ; \tag{14.5}$$

$$|T(t)f| \leq \text{const } G(bt)|f| \quad (t \geq 0 \ , \ f \in L^2(\Omega)) \ . \tag{14.6}$$

From this follows

$$\|T(t)\|_{\mathcal{L}(L^1, L^2)} \leq \text{const} \quad t^{-n/4} \quad (t > 0) \ ; \tag{14.7}$$

$$\|T(t)\|_{\mathcal{L}(L^2, L^\infty)} \leq \text{const} \quad t^{-n/4} \quad (t > 0) \ , \tag{14.8}$$

see Section 12.1. Choose $\delta \in (0,1)$ such that $\delta t + is \in \Sigma(\theta_3)$ whenever $t + is \in \Sigma(\theta_2)$. Let $z = t + is \in \Sigma(\theta_2)$. Then by (14.5), (14.8), (14.9), $\|T(z)\|_{\mathcal{L}(L^1,L^\infty)} \leq \|T((1-\delta)t/2\|_{\mathcal{L}(L^1,L^2)}\|T(\delta t+is)\|_{\mathcal{L}(L^2)} \cdot \|T((1-\delta)t/2)\|_{\mathcal{L}(L^1,L^\infty)} \leq$ const $t^{-n/2} =$ const $(\mathrm{Re} z)^{-n/2}$.

By Proposition 14.2.4 there exists $k : \Sigma(\theta_2)\times\Omega\times\Omega \to \mathbb{C}$ such that $k(\cdot, x, y)$ is holomorphic for all $x, y \in \Omega$ , $k(z, \cdot, \cdot) \in L^\infty(\Omega \times \Omega)$ and $(T(z)f)(x) = \int\limits_\Omega k(z, x, y)f(y)dy \quad (f \in L^1 \cap L^2)$. Moreover,

$$(14.9) \qquad |k(z, x, y)| \leq \text{ const } (\mathrm{Re} z)^{-n/2} \quad (z \in \Sigma(\theta_2))$$

and by (4.5),

$$(14.10) \qquad |k(t, x, y)| \leq \text{ const } t^{-n/2} \exp(-b^{-1}|x - y|^2/t) \quad (t > 0) \ .$$

Applying Lemma 14.2.6 to the function $z^{n/2}k(z, x, y)$ we obtain a constant $c > 0$ such that for $b_1 = b^{-1} \cdot \frac{\sin(\gamma-\theta_1)}{\sin\gamma}$

$$(14.11) \qquad |k(z, x, y)| \leq c \cdot |z|^{-n/2} \exp(-b_1|x - y|^2/|z|)$$

for all $z \in \bar{\Sigma}(\theta_1)$ and all $x, y \in \Omega$. It follows from Lemma 14.2.7 that

$$\sup_{z\in\Sigma(\theta_1)} \|T(z)\|_{\mathcal{L}(L^p)} < \infty \ , \ 1 \leq p < \infty \, .$$

Thus there exist operators $T_p(z) \in \mathcal{L}(L^p)$ such that $T_p(z)f = T(z)f \quad (f \in L^p \cap L^2)$ $(z \in \Sigma(\theta_1))$. It follows from Theorem 14.2.3 that $T_p(\cdot) : \Sigma(\theta_1) \to \mathcal{L}(L^p(\Omega))$ is holomorphic. This finishes the proof. □

We apply the result to the examples we had seen before.

**Example 14.2.8** (Dirichlet Laplacian on $L^1$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. The Dirichlet Laplacian $\Delta^D_{\Omega,p}$ generates a holomorphic $C_0$-semigroup on $L^p(\Omega), 1 \leq p < \infty$. Here $(e^{t\Delta^D_{\Omega,p}})_{t\geq 0}$ is the extrapolated semigroup of $(e^{t\Delta^D_\Omega})_{t\geq 0}$.*

**Example 14.2.9** (elliptic operators with unbounded drift on $L^1$)**.** *Let $\Omega \subset \mathbb{R}^n$. Consider the elliptic operator $A$ on $L^2(\Omega)$ defined in Section 13.2. Then by Theorem 13.2.1 the semigroup $(e^{-tA})_{t\geq 0}$ has an upper Gaussian bound. Thus the extrapolated $C_0$-semigroups $(e^{-tA_p})_{t\geq 0}$ on $L^p(\Omega), 1 \leq p < \infty$ are all holomorphic.*

## 14.3 Convergence to an equilibrium

In this section we consider an elliptic operator with Neumann boundary conditions which generates a stochastic $C_0$-semigroup on $L^1(\Omega)$. We will use the Perron-Frobenius Theory of Lecture 10 to describe its asymptotic behaviour. Throughout this section we consider $\mathbb{K} = \mathbb{R}$ as underlying field. Let $\Omega \subset \mathbb{R}^n$ be a connected, open, bounded set. We start to describe stochastic semigroups on $L^1(\Omega)$.

**Definition 14.3.1.** *A $C_0$-semigroup $T$ on $L^1(\Omega)$ is* **stochastic** *if $T$ is positive and $\|T(t)f\|_{L^1} = \|f\|_{L^1}$ for all $0 \leq f \in L^1(\Omega)$.*

It is easy to describe when a semigroup on $L^2(\Omega)$ associated with an elliptic form admits a stochastic extension. Recall that $L^2(\Omega) \subset L^1(\Omega)$ since we assume here that $\Omega$ is bounded.

**Proposition 14.3.2.** *Let $(a, V)$ be an elliptic, continuous form on $L^2(\Omega)$ such that the associated semigroup $T$ on $L^2(\Omega)$ is positive. Denote by $-A$ the generator of $T$.*
*a) The following assertions are equivalent.*

*(i) $T$ is stochastic,*

*(ii) $1_\Omega \in \ker(A')$,*

*(iii) $1_\Omega \in V$ and $a(u, 1_\Omega) = 0$ for all $u \in V$.*

*b) If the equivalent conditions are satisfied then there exists a unique stochastic $C_0$-semigroup $T_1$ on $L^1(\Omega)$ such that*

$$T_1(t)_{|_{L^2(\Omega)}} = T(t) \quad (t > 0) \ .$$

We leave the proof as an exercise.

Now we consider an elliptic operator. Let $a_{ij} \in L^\infty(\Omega)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad (\xi \in \mathbb{R}^n)$$

a.e. Let $b_j, c_j, c_0 \in L^\infty(\Omega), j = 1, \cdots, n$. Define the form $a$ with domain $V = H^1(\Omega)$ by

$$\begin{aligned} a(u,v) &= a_0(u,v) + \int \{\sum_{j=1}^{n}(b_j D_j uv + c_j u D_j v) + c_0 uv\}dx \\ \text{where } a_0(u,v) &= \int \sum_{i,j=1}^{n} a_{ij} D_i u D_j v dx \ . \end{aligned}$$

**Proposition 14.3.3.** *The form $(a, V)$ is continuous and elliptic and the associated $C_0$-semigroup $T$ is positive and irreducible.*

*Proof.* Continuity is obvious. Ellipticity is proved with the help of Cauchy's inequality

$$a \cdot b = \sqrt{\varepsilon}a \cdot \frac{1}{\sqrt{\varepsilon}}b \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \ .$$

Let $|b| = (\sum\limits_{j=1}^{n} |b_j|^2)^{\frac{1}{2}}$. Then

$$\begin{aligned}\int\limits_{\Omega} \sum_{j=1}^{n} b_j D_j u \cdot u \geq & - \| \, |b| \, \|_\infty \int\limits_{\Omega} |\nabla u| \, |u| dx \\ \geq -\varepsilon \int\limits_{\Omega} |\nabla u|^2 dx & - \frac{1}{\varepsilon} \| \, |b| \, \|_\infty^2 \int\limits_{\Omega} |u|^2 dx \ .\end{aligned}$$

Similarly,

$$\int\limits_{\Omega} \sum_{j=1}^{n} c_j u D_j u \geq -\varepsilon \int\limits_{\Omega} |\nabla u|^2 dx - \frac{1}{\varepsilon} \| \, |c| \, \|_\infty^2 \cdot \int\limits_{\Omega} |u|^2 dx \ .$$

Since $a_0(u) \geq \alpha \int\limits_{\Omega} |\nabla u|^2 dx$, letting $\varepsilon = \frac{\alpha}{3}$ , $\omega = \frac{1}{\varepsilon}(\| \, |b| \, \|_\infty^2 + \| \, |c| \, \|_\infty^2)$ we conclude that

$$\begin{aligned} a(u) & \geq a_0(u) - \frac{2\alpha}{3} \int |\nabla u|^2 - \omega \|u\|_{L^2}^2 \\ & \geq \frac{\alpha}{3} \int\limits_{\Omega} |\nabla u|^2 dx - \omega \|u\|_{L^2}^2 \ .\end{aligned}$$

Hence

$$a(u) + (\omega + \frac{\alpha}{3}) \|u\|_{L^2}^2 \geq \frac{\alpha}{3} \|u\|_{H^1}^2$$

for all $u \in H^1(\Omega)$, which proves ellipticity. Since $a(u^+, u^-) = 0$ for all $u \in H^1(\Omega)$, it follows from the first Beurling-Deny criterion, Theorem 9.2.1, that $T$ is positive. It follows from Theorem 10.1.5 that $T$ is irreducible. $\square$

By

$$W^{1,\infty}(\Omega) := \{u \in H^1(\Omega) \cap L^\infty(\Omega) : D_j u \in L^\infty(\Omega)\}$$

we denote the Sobolev space with norm

$$\|u\|_{W^{1,\infty}} := \max(\{\|D_j u\|_{L^\infty}, j = 1 \cdots n\} \cup \{\|u\|_{L^\infty}\}) \ .$$

Let $W_0^{1,\infty}$ be the closure of $\mathcal{D}(\Omega)$ in $W^{1,\infty}$. Then the following product rule holds.

**Lemma 14.3.4.** *Let $u \in H^1(\Omega), v \in W_0^{1,\infty}(\Omega)$. Then $uv \in H_0^1(\omega)$ and*

$$D_j(uv) = D_j u \cdot v + u D_j v \ . \tag{14.12}$$

*Proof.* For $v \in \mathcal{D}(\Omega)$, (14.13) is true. So it suffices to pass to the limit. $\square$

Now we add the assumption that

$$b_j \in W_0^{1,\infty}(\Omega) \quad \text{and} \quad c_0 = \sum_{j=1}^{n} D_j b_j \ . \tag{14.13}$$

**Proposition 14.3.5.** *Under the assumption (14.14) there exists a unique stochastic $C_0$-semigroup $T_1$ on $L^1(\Omega)$ such that*

$$T_1(t)_{|_{L^2(\Omega)}} = T(t) \quad (t \geq 0) \ .$$

*Proof.* Let $u \in H^1(\Omega)$. Then

$$\begin{aligned} a(u, 1_\Omega) &= \int_\Omega \{\sum_{j=1}^{n} b_j D_j u + c_0 u\} dx \\ &= \int_\Omega \sum_{j=1}^{n} D_j(b_j u) + c_0 u dx \\ & \quad \int_\Omega (-\sum_{j=1}^{n} D_j b_j + c_0) u dx \\ &= \int_\Omega \sum_{j=1}^{n} D_j(b_j u) dx = 0 \end{aligned}$$

since $b_j u \in H_0^1(\Omega), j = 1, \cdots, n$. Now the claim follows from Proposition 14.3.2. □

Now we formulate the convergence result.

**Theorem 14.3.6.** *Assume that $\Omega$ has Lipschitz boundary and that (14.14) holds. Then there exists $0 \ll w \in L^1(\Omega)$ such that*

$$\lim_{t \to \infty} T_1(t) = P \ \text{in} \ \mathcal{L}(L^1(\Omega))$$

*where $Pf = \int_\Omega f dx w$ for all $f \in L^1(\Omega)$. Moreover, $\int_\Omega w dx = 1$.*

For the proof we use the following modification of Theorem 12.1.4 which is proved exactly in the same way (only the first step is needed).

**Proposition 14.3.7.** *Let $S$ be a $C_0$-semigroup on $L^1(\Omega), 2 < q \leq \infty$, such that*

$$\|S(t)\|_{\mathcal{L}(L^2)} \leq c \quad (0 < t \leq 1), \ \text{and} \tag{14.14}$$

$$\|S(t)\|_{\mathcal{L}(L^2, L^q)} \leq c t^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{q})} \quad (0 < t \leq 1) \ . \tag{14.15}$$

*Then there exists a constant $c_1 > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(L^1, L^q))} \leq c_1 t^{-\frac{n}{2}(1 - \frac{1}{q})} \quad (0 < t \leq 1) \ . \tag{14.16}$$

*Proof of Theorem 14.3.6* a) We first prove that $T_1(t)$ is compact for all $t > 0$. For this we need the assumption that $\Omega$ has Lipschitz boundary. In fact, this implies that there exists $2 < q \le \infty$ such that $H^1(\Omega) \subset L^q(\Omega)$ $\left(\frac{1}{q} = \frac{1}{n} - \frac{1}{2}\right.$ if $n > 2; 2 < q < \infty$ arbitrary if $n \le 2)$. Hence $\|u\|_{L^q} \le c_q \|u\|_{H^1}$ for all $u \in H^1(\Omega)$ and some constant $c_q > 0$. Recall from (7.8) that

$$\|T_2(t)\|_{\mathcal{L}(L^2,H^1)} \le c_2 t^{-1/2} \quad (0 < t < 1)$$

for some $c_2 > 0$. It follows that

$$\|T_2(t)\|_{\mathcal{L}(L^2,L^q)} \le c_q c_2 t^{-1/2} \quad (0 < t \le 1) \ .$$

Thus Proposition 14.3.7 implies that $T_1(t)L^1(\Omega) \subset L^2(\Omega)$ for all $t > 0$. Since the injection $j : H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the operator $T_2(t)$ is compact. Factorizing

$$T_1(t) = j \circ T_2(t/2) T_1(t/2)$$

one sees that $T_1(t)$ is compact for all $t > 0$.
b) Denote by $B_1$ the generator of $T_1$. Since $T_1$ is stochastic, it follows that $s(B_1) = 0$, where $s(B_1)$ denotes the spectral bound. It follows from a) that $T_1$ is immediately norm continuous and $B_1$ has compact resolvent (Proposition 2.5.7). Since $T_1$ is immediately norm continuous, $s(B_1) + i\mathbb{R}$ is bounded (see [Nag86, A-II, Theorem 1.20, p. 38] or [EN00, II.4.18, p. 113]). Since $T_2$ is irreducible also $T_1$ is irreducible (this follows directly from the definition). Now since $B_1$ has compact resolvent and $T_1$ is immediately norm continuous it follows from the proof of Theorem 10.4.1 that $\lim\limits_{t\to\infty} T_1(t) = P$ in $L^1(\Omega)$ where $Pf = \varphi(f)w$ for some $0 \ll \varphi \in L^1(\Omega)'$ , $0 \ll w \in L^1(\Omega)$. Since $1_\Omega \in \ker B_1'$ and $\dim \ker B_1' = 1$, it follows that $\varphi(f) = c \int\limits_\Omega f dx$ for all $f \in L^1(\Omega)$ and some $c > 0$. Replacing $w$ by $\frac{w}{c}$ the claim is proved. □

## 14.4 Exercises

In the first exercise we show that ultracontractivity alone allows one to deduce that the induced semigroups on $L^p$ have holomorphic extensions to a sector whenever the given semigroup on $L^2$ is holomorphic.

**Exercise 14.4.1.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $T$ be a holomorphic $C_0$-semigroup on $L^2(\Omega)$ of angle $\theta \in (0, \pi/2]$ such that*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le ct^{-n/2} e^{\omega t} (0 < t \le 1) \ .$$

*Assume that*

$$\begin{aligned} \|T(t)\|_{\mathcal{L}(L^1)} &\le c \quad (0 < t \le 1) \\ \|T(t)\|_{\mathcal{L}(L^\infty)} &\le c \quad (0 < t \le 1) \ . \end{aligned}$$

*a) Observe that there are consistent $C_0$-semigroups $T_p$ on $L^p(\Omega), 1 \le p < \infty$, such that $T_2 = T$.*
*b) Show that for each $1 \le p < \infty$ the semigroup $T_p$ has a holomorphic extension $\Sigma_\theta \to \mathcal{L}(L^p)$.*

**Remark 14.4.2.** *However, it is not clear that*

$$\sup_{\substack{z\in\Sigma'_\theta \\ |z|\le 1}} \|T_p(t)\|_{\mathcal{L}(L^p)} < \infty$$

*for* $0 < \theta' < \theta, 1 \le p < \infty, p \ne 2$. *The point in this lecture was that such an estimate holds if* $T$ *has Gaussian upper bounds.*

Next we give an alternative proof of the fact that the semigroup generated by $\Delta_0$ is holomorphic (Theorem 5.2.4). It uses the fact that the semigroup generated by the Dirichlet Laplacian on $L^1(\Omega)$ is holomorphic (Example 14.2.8) which we proved with the help of Gaussian estimates.

**Exercise 14.4.3** (the Dirichlet Laplacian on $C_0(\Omega)$ revisited)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set which is Dirichlet regular. Consider the Laplacian* $\Delta_0$ *on* $C_0(\Omega)$ *given by*

$$\begin{aligned} D(\Delta_0) &= \{u \in C_0(\Omega : \Delta u \in C_0(\Omega)\} \\ \Delta_0 u &= \Delta u \ . \end{aligned}$$

*Denote by* $\Delta^D_{\Omega,p}$ *the Dirichlet Laplacian on* $L^p(\Omega), 1 \le p < \infty$. *Recall from the proof of Proposition 5.2.8 that* $R(\lambda, \Delta^D_{\Omega,2})C_0(\Omega) \subset C_0(\Omega)$ *for* $\lambda > 0$ *and* $R(\lambda, \Delta_0) = R(\lambda, \Delta^D_{\Omega,2})_{|_{C_0(\Omega)}}$. *Use Example 4.2.8 to show that* $\Delta_0$ *generates a holomorphic* $C_0$*-semigroup.*

**Hint:** *Show that* $R(\lambda, \Delta_0)'_{|_{L^1}} = R(\lambda, \Delta^D_{\Omega,1})$ $(\lambda > 0)$.

**Exercise 14.4.4.** *Prove Proposition 14.3.2*

In the last exercise it is shown that a very weak form of ultracontractivity implies compactness of a semigroup defined on $L^1$. It is an immediate consequence of the following classical result.

**Theorem 14.4.5.** *Let* $B \in \mathcal{L}(L^1)$ *be a weakly compact operator. Then* $B^2$ *is compact.*

We refer to [Sch74] for the proof.

**Exercise 14.4.6.** *Let* $|\Omega| < \infty$ *and let* $T$ *be a* $C_0$*-semigroup on* $L^1(\Omega)$ *such that* $T(t)L^1 \subset L^q$ *for some* $q > 1$ *and all* $t > 0$. *Show that* $T(t)$ *is compact for all* $t > 0$.

# 14.5 Comments

The weak characterization of vector-valued holomorphic functions, Theorem 14.2.3, can be found in [ABHN01, Theorem A7]. It is a consequence of the Krein-Smulyan Theorem, see [AN00], [AN06] for the proof and further information on vector-valued holomorphic functions.
Theorem 14.2.1 is due to Ouhabaz [Ouh92a] in the symmetric case. The extension to the non-symmetric case is given in [AtE97] and Hieber [Hie96]. Here we follow [Are97]. The harmonic oscillator leads to an example of a consistent family of $C_0$-semigroups $T_p$ on $L^p, 1 \le p < \infty$ such that $T_p$ is holomorphic for $1 < p < \infty$, but $T_1$ is not (and not even eventually norm-continuous), see [Dav89, Theorem 4.3.6].

# Lecture 15

# Interpolation of the Spectrum

In this lecture we study spectral properties of generators of semigroups which operate in a consistent way on $L^p$ spaces, $1 \le p < \infty$. Then it may happen that the spectrum of the generator does depend on $p$. We show this by a simple example. The reason for this strange phenomenon is that even though the semigroups are consistent, the resolvents need not to be so for some $\lambda$. We do study this consistency problem giving several illustrating results and examples. The main result given in Section 15.5 shows that the spectrum is $L^p$-independent if the semigroup has a Gaussian upper bound. We start by studying a notion of convergence of a sequence of unbounded operators. There are 5 sections in this lecture.

15.1 Convergence of a sequence of unbounded operators in the resolvent sense.

15.2 Spectral independence with respect to subspaces.

15.3 Consistency of the resolvent.

15.4 Examples.

15.5 $L^p$-independence of the spectrum.

## 15.1 Convergence of a sequence of unbounded operators in the resolvent sense

Here we prove a general result on the continuity of the resolvent. Let $X$ be a complex Banach space.

**Theorem 15.1.1.** *Let $A$ be an operator on $X$ and let $K \subset \rho(A)$ be a compact set. Let $\lambda_0 \in \rho(A)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that the following holds. If $B$ is an* *operator such that* $\lambda_0 \in \rho(B)$ *and*

$$\|R(\lambda_0, A) - R(\lambda_0, B)\|_{\mathcal{L}(X)} \leq \delta,$$

*then* $K \subset \rho(B)$ *and*

$$\|R(\lambda, A) - R(\lambda, B)\|_{\mathcal{L}(X)} \leq \epsilon$$

*for all* $\lambda \in K$.

*Proof.* We may assume that $\lambda_0 = 0$, replacing $A$ by $A - \lambda_0$ and $B$ by $B - \lambda_0$ otherwise. Let $M = \sup_{\mu \in K} \|\mu - \mu^2 R(\mu, A)\|$ and $\delta_0 := \frac{1}{2M}$. Let $0 < \delta < \delta_0$ and $B$ be an operator on $X$ such that $0 \in \rho(B)$ and $\|A^{-1} - B^{-1}\|_{\mathcal{L}(X)} \leq \delta$. Let $\mu \in K \setminus \{0\}$. Then

$$(\mu^{-1} - A^{-1})^{-1} = [\mu^{-1}(A - \mu)A^{-1}]^{-1} = -\mu A R(\mu, A) = \mu - \mu^2 R(\mu, A).$$

Hence $\|(\mu^{-1} - A^{-1})^{-1}(B^{-1} - A^{-1})\| \leq M\delta$. Thus

$$Q := (I - (\mu^{-1} - A^{-1})^{-1}(B^{-1} - A^{-1}))$$

is invertible and

$$Q^{-1} = \sum_{k=0}^{\infty} ((\mu^{-1} - A^{-1})^{-1}(B^{-1} - A^{-1}))^k.$$

Consequently,

$$\begin{aligned}
(\mu - B) &= -\mu(\mu^{-1} - B^{-1})B \\
&= -\mu(\mu^{-1} - A^{-1} + A^{-1} - B^{-1})B \\
&= -\mu(\mu^{-1} - A)^{-1})\{I - (\mu^{-1} - A^{-1})^{-1}(A^{-1} - B^{-1})\}B
\end{aligned}$$

is invertible and $R(\mu, B) = -B^{-1}Q^{-1}(I - \mu R(\mu, A))$. Since

$$R(\mu, A) = -A^{-1}(I - \mu R(\mu, A)),$$

it follows that

$$\begin{aligned}
R(\mu, B) - R(\mu, A) &= (A^{-1} - B^{-1}Q^{-1})(I - \mu R(\mu, A)) \\
&= \left( (A^{-1} - B^{-1}) + \sum_{k=1}^{\infty} \left( (\mu^{-1} - A^{-1})^{-1}(B^{-1} - A^{-1}) \right)^k \right) (I - \mu R(\mu, A)).
\end{aligned}$$

Thus for $c = \sup_{\mu \in K} \|I - \mu R(\mu, A)\|$ we have

$$\|R(\mu, B) - R(\mu, A)\| \leq \delta + \sum_{k=1}^{\infty} (M\delta)^k \cdot c = \delta + \frac{M\delta}{1 - M\delta} \cdot c \leq \delta(1 + 2Mc),$$

since $M\delta \leq \frac{1}{2}$. $\square$

The preceding result leads us to the following definition which we only use to illustrate further the result.

Let $A_n$ be operators on $X, n \in \mathbb{N} \cup \{\infty\}$. We say that $A_n$ **converges to** $A_\infty$ **in the resolvent sense** if there exists $\lambda_0 \in \rho(A_n)$ for all $n \in \mathbb{N} \cup \{\infty\}$ such that

$$\lim_{n\to\infty} \|R(\lambda_0, A_n) - R(\lambda_0, A_\infty)\| = 0.$$

As a consequence of Theorem 15.1.1, for each compact set $K \subset \rho(A_\infty)$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \rho(A_n)$ for all $n \geq n_0$ and

$$\sup_{\lambda\in K} \|R(\lambda, A_n) - R(\lambda, A_\infty)\| \to 0 \quad \text{as} \quad n \to \infty.$$

If $A_n \in \mathcal{L}(X)$ for all $n \in \mathbb{N} \cup \{\infty\}$, then it is easy to see that $A_n$ converges to $A$ in the resolvent sense if and only if $\lim\limits_{n\to\infty} \|A_n - A_\infty\| = 0$. For such a sequence of bounded operators, Theorem 15.1.1 implies the upper semicontinuity of the spectrum.

**Corollary 15.1.2.** *Let $A_n \in \mathcal{L}(X),\ n \in \mathbb{N} \cup \{\infty\}$ such that*

$$\lim_{n\to\infty} A_n = A_\infty \quad \textit{in} \quad \mathcal{L}(X).$$

*Let $O \subset \mathbb{C}$ be open such that $\sigma(A_\infty) \subset O$. Then there exists $n_0 \in \mathbb{N}$ such that $\sigma(A_n) \subset)$ for all $n \geq n_0$.*

## 15.2 Spectral independence with respect to subspaces

In this short introductory section we present a simple result of spectral independence. Let $X, Y$ be Banach spaces such that $Y \hookrightarrow X$ (by this we mean that $Y$ is a subspace of $X$ and the inclusion is continuous). Let $A$ be an operator on $X$. We denote by $A_Y$ **the part of** $A$ **in** $Y$, i.e. $A_Y$ is given by $D(A_Y) = \{x \in D(A) \cap Y : Ax \in Y\},\ A_Y x = Ax$.

**Proposition 15.2.1.** *Assume that there exists $\mu \in \rho(A)$ such that $R(\mu, A)Y \subset Y$ and that there exists $k \in \mathbb{N}$ such that $D(A^k) \subset Y$. Then $\sigma(A) = \sigma(A_Y)$ and $R(\lambda, A_Y) = R(\lambda, A)_{|Y}$ for all $\lambda \in \rho(A)$.*

*Proof.* a) Let $\lambda \in \rho(A)$. Iteration of the resolvent equation $R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)$ yields

$$R(\lambda, A) = \sum_{j=1}^{k} (\mu - \lambda)^{j-1} R(\mu, A)^j + (\mu - \lambda)^k R(\mu, A)^k R(\lambda, A). \tag{15.1}$$

This shows that $R(\lambda, A)Y \subset Y$. It is now obvious that $\lambda \in \rho(A_Y)$ and $R(\lambda, A_Y) = R(\lambda, A)_{|Y}$.

b) Conversely, let $\lambda \in \rho(A_Y)$. The space $D(A^k)$ is a Banach space for the norm $\|x\|_{D(A^k)} = \|(\mu - A)^k x\|_X$ and $D(A^k) \hookrightarrow X$. Since $Y \hookrightarrow X$ it follows from the closed graph theorem that $D(A^k) \hookrightarrow Y$. Note that $R(\mu, A)^k$ is anisomorphism of $X$ onto $D(A^k)$. Thus

$$Qx := \sum_{j=1}^{k} (\mu - \lambda)^{j-1} R(\mu, A)^j x + (\mu - \lambda)^k R(\lambda, A_Y) R(\mu, A)^k x \qquad (x \in X)$$

defines a bounded operator on $X$. Moreover, for $x \in X, Qx \in D(A)$ and

$$(\lambda - A)Qx = \sum_{j=1}^{k} \{(\mu - \lambda)^{j-1} R(\mu, A)^{j-1} x - (\mu - \lambda)^j R(\mu, A)^j x\} + (\mu - \lambda)^k R(\mu, A)^k x = x.$$

Since for $x \in D(A), AQx = QAx$, it follows that $\lambda \in \rho(A)$ and $Q = R(\lambda, A)$. □

## 15.3 Consistency of the resolvent

Let $X, Y$ be two Banach spaces. We assume that there exists a third Banach space $Z$ such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z$.

**Definition 15.3.1.** *Two operators $B_X \in \mathcal{L}(X)$ and $B_Y \in \mathcal{L}(Y)$ are* **consistent** *if*

$$B_X x = B_Y x \qquad (x \in X \cap Y).$$

Let $T_X$ and $T_Y$ be $C_0$-semigroups on $X$ and $Y$, resp., with generators $A_X$ and $A_Y$, resp. We assume that $T_X$ and $T_Y$ are **consistent**, i.e., that $T_X(t)$ and $T_Y(t)$ are consistent for all $t \geq 0$. We will see below (Section 15.4) that this does not imply in general that $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent for all $\lambda \in \rho(A_X) \cap \rho(A_Y)$.

**Proposition 15.3.2.** *The set $\mathcal{U}$ of all $\lambda \in \rho(A_X) \cap \rho(A_Y)$ such that $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent is open and closed in $\rho(A_X) \cap \rho(A_Y)$.*

Note that $X + Y$ is Banach space for the norm

$$\|u\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : x \in X, y \in Y, u = x + y\}$$

and $X \cap Y$ is a Banach space for the norm

$$\|u\|_{X \cap Y} = \|u\|_X + \|v\|_Y.$$

The injections $X \cap Y \hookrightarrow X \hookrightarrow X + Y, X \cap Y \hookrightarrow Y \hookrightarrow X + Y$ are continuous. In particular, if $x_n \in X \cap Y, x_n \to x$ in $X$ and $x_n \to y$ in $Y$, then $x = y$ and $x_n \to x$ in $X \cap Y$.

*Proof of Proposition 15.3.2.* It follows from the remark above that $\mathcal{U}$ is closed in $\rho(A_X) \cap \rho(A_Y)$. Let $\lambda_0 \in \mathcal{U}$. Let $\epsilon > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(A_Y)$. Then for $x \in X \cap Y$, $R(\lambda, A_X)x = \sum\limits_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A_X)^{n+1} x$ and $R(\lambda, A_Y)x = \sum\limits_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A_Y)^{n+1}$ where $|\lambda - \lambda_0| < \epsilon$. Since $R(\lambda_0, A_X))^{n+1}$ and $R(\lambda, A_Y)^{n+1}$ are consistent, it follows from the remark above that $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent. $\square$

Recall that for $x, y \in X$,

$$\begin{aligned} &x \in D(A_X),\ A_X x = y \qquad \text{if and only if} \\ &\int_0^t T_X(s) y ds = T_X(t)x - x \qquad (t \geq 0). \end{aligned} \tag{15.2}$$

In the following we assume that $X \cap Y$ is dense in $X$ and in $Y$.

**Proposition 15.3.3.** *Let $\lambda \in \rho(A_X)$. Assume that there exists $Q \in \mathcal{L}(Y)$ which is consistent with $R(\lambda, A_X)$. Then $\lambda \in \rho(A_Y)$ and $R(\lambda, A_Y) = Q$.*

*Proof.* We can assume that $\lambda = 0$ (considering $A_X - \lambda$ otherwise). It follows from (15.2) that

$$\int_0^t T_X(s) y ds = T_X(t) A_X^{-1} y - A_X^{-1} y \qquad (y \in X, t \geq 0).$$

Hence $\int_0^t T_Y(s) y ds = T_Y(t) Q y$ $((t \geq 0)$ for all $y \in Z \cap Y$, and by density, for all $y \in Y$. It follows from (15.2) (with $X$ replaced by $Y$) that $Qy \in D(A_Y)$ and $A_Y Q y = y$ for all $y \in Y$. Since $Q T_Y(t) y = T_Y(t) Q y$ if $y \in X \cap Y$, it follows that $Q$ and $T_Y(t)$ commute $(t \geq 0)$. Hence $A_Y Q y = Q A_Y y$ if $y \in D(A_Y)$. $\square$

The following is a converse of Proposition 15.3.3.

**Proposition 15.3.4.** *Let $\lambda \in \rho(A_X) \cap \rho(A_Y)$. If $R(\lambda, A_T(X \cap Y) \subset X \cap Y$, then $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent.*

*Proof.* We can assume that $\lambda = 0$. Let $x \in X \cap Y$. By hypothesis $A_Y^{-1} x \in X \cap Y$. Hence

$$\int_0^t T_X(s) x ds = \int_0^t T_Y(s) x ds = T_Y(t) A_Y^{-1} x - A_Y^{-1} x = T_X(t) A_Y^{-1} x - A_Y^{-1} x \qquad (t \geq 0).$$

It follows from (15.2) that $A_Y^{-1} x \in D(A_X)$ and $A_X(A_Y^{-1} x) = x$; i.e. $A_Y^{-1} x = A_X^{-1} x$. $\square$

**Corollary 15.3.5.** *Assume that $Y \subset X$. Then $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent for all $\lambda \in \rho(A_X) \cap \rho(A_Y)$.*

**Proposition 15.3.6.** *Assume that*

*(a)* $T_X(t)X \subset Y$ *for some* $t > 0$ *or*

*(b)* $D(A_X^k) \subset Y$ *for some* $k \in \mathbb{N}$.

*Then* $R(\lambda, A_X)$ *and* $R(\lambda, A_Y)$ *are consistent for all* $\lambda \in \rho(A_X) \cap \rho(A_Y)$.

*Proof.* a) Let $\lambda \in \rho(A_X) \cap \rho(A_Y)$. We can assume that $\lambda = 0$. Let $x \in X \cap Y$. Then by (15.2), $A_X^{-1}x = T_X(t)A_X^{-1} - \int_0^t T_X(s)xds = T_X(t)A_X^{-1}x - \int_0^t T_Y(s)xds \in X \cap Y$. It follows from Proposition 15.2.4 that $A_X^{-1}$ and $A_Y^{-1}$ are consistent.
b) If $\mu$ is larger than the type of $T_X$ and $T_Y$, then $R(\mu, A_X)$ and $R(\mu, A_Y)$ are consistent since they are the Laplace transforms of the consistent semigroups. Let $\lambda \in \rho(A_X) \cap \rho(A_Y)$. It follows from (15.2) that $R(\lambda, A_X)(X \cap Y) \subset X \cap Y$. Thus the claim follows from Proposition 15.2.4. □

**Proposition 15.3.7.** *Assume that* $A_X$ *and* $A_Y$ *have compact resolvent. Then* $\sigma(A_X) = \sigma(A_Y)$.

*Proof.* Since $\rho(A_X) \cap \rho(A_Y)$ is connected, $R(\mu, A_X)$ and $R(\mu, A_Y)$ are consistent for all $\mu \in \rho(A_X) \cap \rho(A_Y)$. Let $\lambda_0 \in \rho(A_Y)$. Since $\sigma(A_X)$ consists of isolated points, there exists $\epsilon > 0$ such that $\{\lambda \in \mathbb{C} : 0 < |\lambda - \lambda_0| \le \epsilon\} \subset \rho(A_X) \cap \rho(A_Y)$. Since $\lambda_0 \in \rho(A_Y)$, one has

$$\int_{|\lambda-\lambda_0|=\epsilon} R(\lambda, A_Y)d\lambda = 0.$$

By consistency, it follows that

$$\int_{|\lambda-\lambda_0|=\epsilon} R(\lambda, A_X)d\lambda = 0,$$

hence $\lambda_0 \in \rho(A_X)$. □

## 15.4 Examples

We give three example of consistent operators on $L^p$ whose spectra depend on $p$.

**Example 15.4.1.** *Define the consistent* $C_0$*-groups* $T_p$ *on* $L^p(0, \infty)$ *by*

$$(T_p(t)f)(x) = f(e^{-t}x) \qquad (t \in \mathbb{R}),$$

$1 \le p < \infty$ *and denote by* $A_p$ *the generator of* $T_p$. *Then* *(a)* $\sigma(A_p) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \frac{1}{p}\}$;

*(b)* $R(\lambda, A_p)$ *and* $R(\lambda, A_q)$ *are not consistent whenever* $p < q$ *and* $\frac{1}{q} < \operatorname{Re}\lambda < \frac{1}{p}$;

*(c)* $A_p$ *is given by* $(A_p f)(x) = -xf'(x)$,

$$D(A_p) = \{f \in L^p(0,\infty) : x \mapsto xf'(x) \in L^p(0,\infty)\}.$$

*Proof.* For $f \in L^p(0,\infty)$ one has $\|T_p(t)f\|_p = (\int\limits_0^\infty |f(e^{-t}x)|^p dx)^{\frac{1}{p}} = e^{\frac{1}{p}}|f\|_p$. Hence $(e^{-\frac{1}{p}}T_p(t))_{t\in\mathbb{R}}$ is an isometric group on $L^p(0,\infty)$. It follows that its generator $A_p - \frac{1}{p}$ has spectrum in $i\mathbb{R}$, i.e. $\sigma(A_p) \subset \frac{1}{p} + i\mathbb{R}$. Let $\frac{1}{q} < \lambda < \frac{1}{p}$. Since the type of $T_q$ is $\frac{1}{q}$ and the type of $(T_p(t))_{t\geq 0}$ is $-\frac{1}{p}$ we have $R(\lambda, A_q) = \int\limits_0^\infty e^{\lambda t}T_q(t)dr \geq 0$ and $R(\lambda, A_p) = -R(-\lambda, -A_p) = -\int\limits_0^\infty e^{\lambda t}T_q(-t)dt \leq 0$. Thus $R(\lambda, A_p)$ and $R(\lambda, A_q)$ are not consistent. It follows from Proposition 15.3.2 that $\sigma(A_p) = \frac{1}{p} + i\mathbb{R}, \sigma(A_q) = \frac{1}{q} + i\mathbb{R}$ and that $R(\lambda, A_p)$ and $R(\lambda, A_q)$ are not consistent on the entire strip $\{\lambda \in \mathbb{C} : \frac{1}{q} < \operatorname{Re}\lambda < \frac{1}{p}\}$. We have shown (a) and (b). The last point (c) will become clear from 2. ☐

**Example 15.4.2.** *Let* $C_p f)(x) = \frac{1}{x}\int\limits_0^x f(y)dy, 1 < p < \infty$. *Then* $C_p$ *is a bounded operator on* $L^p(0,\infty)$, $\|C_p\| \leq \frac{p}{p-1}$, *and* $\sigma(C_p) = \{\frac{1}{1-\frac{1}{p}-is} : s \in \mathbb{R}\} \cup \{0\}$, *so that* $\sigma(C_p) \cap \sigma(C_q) = \{0\}$ *if* $1 < p, q < \infty, p \neq q$.

*The norm estimate of* $C_p$ *is known as* **Hardy's inequality**. *We obtain both as an easy consequence of 1.*

*Proof.* Let $1 < p < \infty$. Then by 1., $1 \in \rho(A_p)$ and $(R(1, A_p)f)(x) = \int\limits_0^\infty e^{-t}f(e^{-t}x)dt = \frac{1}{x}\int\limits_0^x f(y)dy$. Hence $C_p = R(1, A_p)$. Since $\|T_p(t)\| = e^{\frac{t}{p}}$ we have

$$\|R(1, A_p)\| \leq \int\limits_0^\infty e^{-t}e^{\frac{1}{p}}dt = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}.$$

Since, by Proposition 1.2.3, $\sigma(R(1, A_p)) = \{\frac{1}{1-\lambda} : \lambda \in \sigma(A_p)\} \cup \{0\}$ the assrtion on the spectrum of $C_p$ follows from Example 15.4.1. Now 1(c) is an immediate consequence of $R(1, A_p) = C_p$. ☐

**Example 15.4.3.** *Let* $B_p = (A_p - \frac{1}{2})^2$. *Then* $B_p$ *generates a holomorphic semigroup since* $A_p - \frac{1}{2}$ *generates a* $C_0$*-group on* $L^p(0,\infty)$ *(see e.g. [ABHN01, Exa. 3.14.15 and*

*Thm. 3.14.17]). The group generated by $A_2 - \frac{1}{2}$ is isometric, thus $B_2$ is self-adjoint. By the spectral mapping theorem one has*

$$\begin{aligned}\sigma(B_2) &= (-\infty, 0]\\ \sigma(B_p) &= \left\{(\frac{1}{p} - \frac{1}{2} + is)^2 : s \in \mathbb{R}\right\}\end{aligned}$$

*($1 \leq p < \infty$). Hence $\sigma(B_p) \cap \sigma(B_q) = \emptyset$ whenever $1 \leq p, q \leq 2, p \neq q$. This follows immediately from Example 15.4.1. It is easy to see that $B_p$ is given by*

$$\begin{aligned}D(B_p) &= \{f \in L^p(0,1) : xf' \in L^p(0,1), x^2 f'' \in L^p(0,\infty)\}\\ (B_p f)(x) &= x^2 f'' + 2xf' + \frac{f}{4}.\end{aligned}$$

*Thus $B_p$ is a degenerate elliptic operator of second order.*

## 15.5 $L^p$-independence of the spectrum

Let $\Omega \subset \mathbb{R}^n$ be open and let $T$ be a $C_0$-semigroup on $L^2(\Omega)$ having an upper Gaussian bound, i.e.,

$$|T(t)f| \leq ce^{\omega t} G(bt)|f| \tag{15.3}$$

for all $f \in L^2(\Omega), t \geq 0$ where $G$ is the Gaussian semigroup and $c, b > 0, \omega \in \mathbb{R}$ (cf. Lecture 13). This means that $T$ is given by a kernel $k(t, \cdot, \cdot) \in L^\infty(\Omega \times \Omega)$ satisfying

$$|k(t,x,y)| \leq ce^{\omega t}(4\pi bt)^{-n/2} e^{-|x-y|^2/4bt} \qquad x, y \text{ - a.e.} \tag{15.4}$$

for all $t > 0$. By Corollary 14.1.2, there exists a consistent family of semigroups $(T_p(t))_{t\geq 0}$ on $L^p(\Omega)$, $1 \leq p \leq \infty$, such that $(T_\infty(t))_{t\geq 0} = (T(t))_{t\geq 0}$. Here $(T_p(t))_{t\geq 0}$ is a $C_0$-semigroup for $1 < p < \infty$ and $(T_\infty(t))_{t\geq 0}$ is a dual semigroup on $L^\infty(\Omega)$. Denote by $A_p$ the generator of $(T_p(t))_{t\geq 0}$. Our aim is to prove the following theorem.

**Theorem 15.5.1.** *One has*

$$\sigma(A_p) = \sigma(A_2) \qquad (1 \leq p \leq \infty).$$

For the proof of Theorem 15.5.1 we may assume that $\omega = 0$ replacing $T(t)$ by $e^{-\omega t}T(t)$ otherwise. Thus, we assume that $\omega = 0$ in the sequel. The proof will be given by considering perturbations of $T_p$ by modifying the kernel by certain weights. Let $w : \Omega \to \mathbb{R}$ be a continuous function such that $w(x) > 0$ for all $x \in \Omega$ and such that

$$\frac{w(x)}{w(y)} \leq e^{\alpha|x-y|} \qquad (x, y \in \Omega) \tag{15.5}$$

where $\omega > 0$. We call such a function a **weight function** in the sequel. Let $1 \leq p < \infty$. The operator

$$\mathcal{U}_{w,p} : L^p(\Omega, w^{-p}dx) \to L^p(\Omega)$$

defined by $\mathcal{U}_{w,p}f = \frac{f}{w}$ is an isometric isomorphism. Thus

$$\tilde{S}_p(t) = \mathcal{U}_{w,p}^{-1} T_p(t) \mathcal{U}_{w,p}$$

defines a $C_0$-semigroup $\tilde{S}_p$ on $L^p(\Omega, w^{-p}dx)$. The operator $\tilde{S}_p(t)$ is given by the kernel

$$k_w(t,x,y) = \frac{w(x)}{w(y)} k(t,x,y). \tag{15.6}$$

We now show that this kernel also defines a $C_0$-semigroup on $L^p(\Omega)$.

**Lemma 15.5.2.** *There exists a $C_0$-semigroup $(S_p(t))_{t\geq 0}$ on $L^p(\Omega)$ such that $S_p(t)$ is given by the kernel $k_w(t,\cdot,\cdot)$. Moreover,*

$$|S_p(t)f| \leq c 2^{n/2} e^{2bt\alpha^2} G(2bt)|f| \tag{15.7}$$

*for all $t \geq 0, f \in L^p(\Omega)$.*

*Proof.* By the assumption (15.5.6) the kernel $k_w$ satisfies

$$|k_w(t,x,y)| \leq c(4\pi bt)^{-n/2} e^{-|x-y|^2/8bt} e^{\alpha|x-y|} \cdot e^{-|x-y|^2/8bt}$$

$x, y$ - a.e. Observe that

$$\sup_{s\geq 0}(-s^2/8bt + \alpha s) = 2bt\alpha^2.$$

Hence

$$|k_w(t,x,y)| \leq c(4\pi bt)^{-n/2} e^{2bt\alpha^2} e^{-|x-y|^2/8bt}$$

$x, y$ - a.e. Thus the kernel $k_w(t,\cdot,\cdot)$ defines a bounded operator $S_p(t)$ on $L^p(\Omega)$ satisfying

$$|S_p(t)f| \leq c 2^{n/2} e^{2bt\alpha^2} G(2bt)|f|$$

for all $t > 0$ and $f \in L^p(\Omega)$. Since $S_p(t)f = \tilde{S}_p(t)f$ for all $f \in L^p(\Omega) \cap L^p(\Omega, w^{-p}dx)$ it follows that $S_p(t+s) = S_p(t)S_p(s)$ for all $s, t \geq 0$. It remains to show that

$$\lim_{t\downarrow 0} \|S_p(t)f - f\|_{L^p(\Omega)} = 0$$

for all $f \in L^p(\Omega)$. It suffices to prove this on a dense subspace of $L^p(\Omega)$. Thus we may assume that $f \in L^p(\Omega)$ vanishes outside of a compact set $K \subset \Omega$. Since $\inf_{x\in K} w(x) > 0$, it follows that

$$\begin{aligned}\left(\int_K |S_p(t)f - f|^p dx\right)^{1/p} &\leq C\|S_p(t)f - f\|_{L^p(\Omega, w^{-p}dx)} \\ &= C\|\tilde{S}_p(t)f - f\|_{L^p(\Omega, w^{-p}dx)} \to 0\end{aligned}$$

for some $C > 0$, as $t \downarrow 0$. Outside of $K$ the function $f$ vanishes, hence by (15.7)

$$
\begin{aligned}
&(\int_{\Omega\setminus K} |S_p(t)f - f|^p dx)^{1/p}\\
&\quad = (\int_{\Omega\setminus K} |S_p(t)f|^p dx)^{1/p}\\
&\quad \le 2^{\frac{n}{2}}(\int_{\Omega\setminus K} (e^{2b\alpha^2 t}G(2bt)|f|)^p dx)^{1/p}\\
&\quad \le 2^{\frac{n}{2}}(\int_{\Omega\setminus K} (e^{2b\alpha^2 t}G(2bt)|f| - |f|)^p dx)^{1/p}\\
&\quad \le 2^{\frac{n}{2}}\|e^{2b\alpha^2 t}G(2bt)|f| - |f|\|_{L^p(\mathbb{R}^n)} \to 0
\end{aligned}
$$

as $t \downarrow 0$. This concludes the proof. ☐

Now for $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{R}^n$ let $w_\epsilon$ be a weight satisfying

$$
\frac{w_\epsilon(x)}{w_\epsilon(y)} \le e^{\alpha_\epsilon|x-y|} \qquad (x, y \in \Omega) \tag{15.8}
$$

where $\alpha_\epsilon > 0$, $\lim\limits_{|\epsilon|\to 0} \alpha_\epsilon = 0$. Let $T_{\epsilon,p}$ be the $C_0$-semigroup defined by the kernel $k_{w_\epsilon}$ on $L^p(\Omega)$ according to Lemma 15.5.2. By $A_{\epsilon,p}$ we denote the generator of $T_{\epsilon,p}$.

**Lemma 15.5.3.** *There exist $\epsilon_0 > 0$, $M_1 > 0$ such that for all $|\epsilon| \le \epsilon_0$, $p \in [1, \infty)$,*

$$
\|T_{\epsilon,p}(t)\| \le M_1 e^t \qquad (t \ge 0). \tag{15.9}
$$

*If $p \in [1, \infty)$ and $\lambda \in \rho(A_p)$, then there exists and $\epsilon_1 \in (0, \epsilon_0]$ such that $\lambda \in \rho(A_{\epsilon,p})$ whenever $|\epsilon| \le \epsilon_1$ and*

$$
\lim_{|\epsilon|\to 0} \|R(\lambda, A_{\epsilon,p}) - R(\lambda, A_p)\| = 0. \tag{15.10}
$$

*Proof.* Since the Gaussian semigroup is contractive, the first assertion follows directly from Lemma 15.5.2. Let $0 < \delta < 1$. We show that $T_{\epsilon,p}(t) \to T_p(t)$ as $|\epsilon| \to 0$ uniformly in $t \in [\delta, 1/\delta]$. In fact, the operator $T_p(t) - T_{\epsilon,p}(t)$ is a kernel operator with kernel

$$
\tilde{k}_\epsilon(t, x, y) := k(t, x, y)(1 - \omega_\epsilon(x)\omega_\epsilon(y)^{-1}).
$$

It follows from (15.8) that

$$
e^{-\alpha_\epsilon|x-y|} \le \frac{w_\epsilon(x)}{w_\epsilon(y)} \le e^{\alpha_\epsilon|x-y|},
$$

hence
$$|1-\frac{w_\epsilon(x)}{w_\epsilon(y)}| \le |1-e^{\alpha_\epsilon|x-y|}| \qquad (x,y-\text{a.e.}).$$
By (15.4) the kernel satisfies $\tilde{k}_\epsilon$ of $T_p(t)-T_{p,\epsilon}(t)$ satisfies
$$|\tilde{k}_\epsilon(t,x,y)| \le c(4\pi bt)^{-n/2}e^{-|x-y|^2/4bt}|1-e^{\alpha_\epsilon|x-y|}|$$
$x,y$-a.e. It follows from Young's inequality (Proposition 15.5.4 below) that
$$\|T_p(t)-T_{\epsilon,p}(t)\|_{\mathcal{L}(L^p(\Omega))} \le c(4\pi bt)^{-n/2}\int\limits_{\mathbb{R}^n} e^{-|x|^2/4bt}|1-e^{\alpha_\epsilon|x|}|dx \to 0$$
as $|\epsilon|\to 0$ uniformly in $t\in[\delta,\delta^{-1}]$. The claim is proved.

Now let $\lambda>1$. Let $\epsilon_0>0, M_1>0$ such that $\|T_{\epsilon,p}(t)\| \le M_1e^t \qquad (t\ge 0)$ for $|\epsilon|\le\epsilon_0$. Then
$$\begin{aligned}
\overline{\lim_{|\epsilon|\to 0}} \quad & \|R(\lambda,A_{p,\epsilon})-R(\lambda,A_p)\|_{L^p(\Omega)}\\
&= \overline{\lim_{|\epsilon|\to 0}}\|\int\limits_0^\infty e^{-\lambda t}(T_{\epsilon,p}(t)-T_p(t))dt\|_{\mathcal{L}(L^p(\Omega))}\\
&\le \overline{\lim_{|\epsilon|\to 0}}(\int\limits_0^\delta+\int\limits_\delta^{1/\delta}+\int\limits_{1/\delta}^\infty)e^{-\lambda t}\|T_{\epsilon,p}(t)-T(t)\|_{\mathcal{L}(L^p)}dt\\
&\le 2M(\delta+\frac{1}{\lambda-1}e^{-(\lambda-1)/\delta}).
\end{aligned}$$
The last expression converges to 0 as $\delta\downarrow 0$. Thus (15.10) is proved if $\lambda>1$. For arbitrary $\lambda\in\rho(A_p)$ assertion (15.10) now follows from Theorem 15.1.1. □

We recall Young's inequality, which was used in the proof of the preceding lemma.

**Proposition 15.5.4.** *Let $h\in L^1(\mathbb{R}^n)$, $1\le p\le\infty$. Let $f\in L^p(\mathbb{R}^n)$. Then $f(x-\cdot)h(\cdot)\in L^1(\mathbb{R}^n)$ for almost all $x\in\mathbb{R}^n$. Moreover there exists a function $f*h\in L^p(\mathbb{R}^n)$ such that*
$$f*h(x)=\int\limits_{\mathbb{R}^n} f(x-y)h(y)dy \qquad (x-\text{a.e.}).$$
*Moreover, one has*
$$\|f*h\|_{L^p} \le \|f\|_{L^p}\|h\|_{L^1} \qquad \text{(Young's inequality)}.$$

After these preparations we prove Theorem 15.5.1.

*Proof of Theorem 15.5.1.* a) Let $1 \le p, q < \infty$. We want to show that $\rho(A_p) = \rho(A_q)$. For $\epsilon, x \in \mathbb{R}^n$ we denote by $\epsilon x = \sum_{j=1}^n \epsilon_j x_j$ the canonical scalar product and we let $\omega_\epsilon(x) := e^{\epsilon x}$. Then $\omega_\epsilon$ satisfies (15.8) for $\epsilon_\epsilon = |\epsilon|$. Let $T_{\epsilon,p}$ be the $C_0$-semigroup defined by the kernel $K_{\omega_\epsilon}$ on $L^p(\Omega)$ according to Lemma 15.3.2. Denote by $A_{\epsilon,p}$ its generator.

Define for $\epsilon \in \mathbb{R}^n$ the space $L^p_\epsilon := L^p(\Omega, e^{-\epsilon p x}dx)$. Then $W_{\epsilon,p}f(x) := e^{-\epsilon x}f(x)$ defines an isometric isomorphism $L^p_\epsilon \to L^p(\Omega)$. Thus, $\tilde{T}_{\epsilon,p}(t) := W_{\epsilon,p}^{-1}T_p(t)W_{\epsilon,p}$ defines a $C_0$-semigroup on $L^p_\epsilon$. Its generator $\tilde{A}_{\epsilon,p}$ is similar to $A_p$ and hence $\sigma(\tilde{A}_{\epsilon,p}) = \sigma(A_p)$. The operator $\tilde{T}_{\epsilon,p}(t)$ is represented by the kernel $k_{w_\epsilon}(t, \cdot)$ and therefore $\tilde{T}_{\epsilon,p}(t)$ is consistent with $T_{\epsilon,p}(t)$ (where $\tilde{T}_{\epsilon,p}(t)$ is defined on $L^p_\epsilon$ and $T_{\epsilon,p}(t)$ on $L^p(\Omega)$). Now let $\lambda \in \rho(A_p)$. We want to show that $\lambda \in \rho(A_q)$. It follows from Proposition 2.2.6 and by rescaling that

$$R(\lambda, A_p) = \int_0^1 e^{-\lambda t}T_p(t)dt + e^{-\lambda}T_p(1)R(\lambda, A_p) \tag{15.11}$$

(see Exercise 15.6.1). By Proposition 15.3.3 it suffices to show that $\|R(\lambda, A_p)\|_{\mathcal{L}(L^q)} < \infty$. The term $\int_0^1 e^{-\lambda t}T_p(t)dt$ is consistent with $\int_0^1 e^{-\lambda t}T_q(t)dt \in \mathcal{L}(L^q(\Omega))$. Thus it suffices to show that

$$\|e^{-\lambda t}T_p(1)R(\lambda, A_p)\|_{\mathcal{L}(L^q)} < \infty. \tag{15.12}$$

By Lemma 15.5.3 there exists $\epsilon_1 > 0$ such that $\lambda \in \rho(A_{\epsilon,p})$ whenever $|\epsilon| \le \epsilon_1$ and

$$\sup_{|\epsilon|\le\epsilon_1} \|R(\lambda, A_{\epsilon,p})\| < \infty.$$

Moreover, by (15.7) we may choose $\epsilon_1 > 0$ so small that

$$\sup_{|\epsilon|\le\epsilon_1} (\|T_{\epsilon,p}(1/2)\|_{\mathcal{L}(L^1,L^p)} + \|T_{\epsilon,p}(\frac{1}{2})\|_{\mathcal{L}(L^p,L^\infty)}) < \infty.$$

Observe that

$$T_{\epsilon,p}(1)R(\lambda, A_{\epsilon,p}) = T_{\epsilon,p}(\frac{1}{2})R(\lambda, A_{\epsilon,p})T_{\epsilon,p}(\frac{1}{2}).$$

It follows that

$$\|T_{\epsilon,p}(1)R(\lambda, A_{\epsilon,p})\|_{\mathcal{L}(L^1,L^\infty)} \le c_1 \tag{15.13}$$

whenever $|\epsilon| \le \epsilon_1$ where $c_1 > 0$ is a constant. By the Dunford-Pettis criterion (Theorem 4.1.1) the operator $T_{\epsilon,p}(1)R(\lambda, A_{\epsilon,p})$ is given by a kernel $k_\epsilon \in L^\infty(\Omega \times \Omega)$ such that

$\|k_\epsilon\|_{L^\infty} \le c_1$ whenever $|\epsilon| \le \epsilon_1$. In particular, $T_p(1)R(\lambda, A_p)$ is given by the kernel $k_0$. We claim that

$$k_\epsilon(x,y) = e^{\epsilon(x-y)}k_0(x,y) \qquad (x,y - \text{a.e.}) \tag{15.14}$$

whenever $|\epsilon| \le \epsilon_1$. Once (15.17) is shown the proof is accomplished as follows. It follows from (15.15) that

$$k_0(x,y) = e^{-\epsilon(x-y)}k_\epsilon(x,y) \qquad (x,y - \text{a.e.})$$

whenever $|\epsilon| \le \epsilon_1$. Observe that $\inf\limits_{|\epsilon|\le\epsilon_1} -\epsilon(x-y) = -\epsilon_1|x-y|$ by the Cauchy-Schwarz inequality. Now it follows from Lemma 13.1.5 that

$$|k_0(x,y)| \le e^{-\epsilon_1|x-y|} \qquad (x,y - \text{a.e.}).$$

This implies that $k_0$ defines a bounded operator on $L^q(\Omega)$ by Young's inequality. Thus (15.15) is shown and it follows that $\lambda \in \rho(A_q)$. We have shown that $\rho(A_p) \subset \rho(A_q)$. Since $1 \le p, q < \infty$ are arbitrary, it follows that $\rho(A_p) = \rho(A_q)$ and the proof of a) is finished, once the claim (15.17) is proved. This means we have to show that $e^{\epsilon(x-y)}k_0(x,y)$ is the kernel of $T_{\epsilon,p}(1)R(1, A_{\epsilon,p})$. It is clear from the definition that $e^{\epsilon(x-y)}k_0(x,y)$ is the kernel of $\tilde{T}_{\epsilon,p}(1)R(\lambda, \tilde{A}_{\epsilon,p})$ on the space $L^p_\epsilon$. We know that $T_{\epsilon,p}(1)$ and $\tilde{T}_{\epsilon,p}(1)$ are consistent. Thus it remains to show that $R(\lambda, A_{\epsilon,p})$ and $R(\lambda, \tilde{A}_{\epsilon,p})$ are consistent for $|\epsilon| \le \epsilon_1$. Recall that $\lambda \in \rho(\tilde{A}_{\epsilon,p})$ since $\tilde{A}_{\epsilon,p}$ and $A_{\epsilon,p}$ are similar. To show consistency we use an auxiliar semigroup. Let $v_\epsilon(x) = e^{\epsilon x - |\epsilon||x|} \quad (x \in \Omega)$ where $\epsilon \in \mathbb{R}^n$. Then $v_\epsilon$ is a weight satisfying (15.8) for $\alpha_\epsilon = 2|\epsilon|$. Define the semigroup $(S_{\epsilon,p}(t))_{t\ge0}$ on $L^p(\Omega)$ associated with the kernel

$$\frac{v_\epsilon(x)}{v_\epsilon(y)}k(t,x,y)$$

according to Lemma 15.3.2. Denote by $B_{\epsilon,p}$ the generator of $S_{\epsilon,p}$. Define the space $\hat{L}^p_\epsilon := L^p(\Omega, e^{-p|\epsilon||x|}dx)$. Then $(V_{\epsilon,p}f)(x) = e^{-|x||\epsilon|}f(x)$ defines an isometric isomorphism $V_{\epsilon,p}$ from $\hat{L}^p_\varepsilon$ onto $L^p(\Omega)$. Define the semigroup $\hat{T}_{\epsilon,p}$ on $\hat{L}^p_\epsilon$ by

$$\hat{T}_{\epsilon,p}(t) := V_{\epsilon,p}^{-1}S_{\epsilon,p}(t)V_{\epsilon,p}$$

and denote by $\hat{A}_{\epsilon,p}$ its generator. Then $\rho(\hat{A}_{\epsilon,p}) = \rho(B_{\epsilon,p})$ by similarity. The operator $\hat{T}_{\epsilon,p}(t)$ is given by the kernel

$$e^{\epsilon(x-y)}k(t,x,y) = \frac{e^{|\epsilon||x|}}{e^{|\epsilon||y|}}\frac{v_\epsilon(x)}{v_\epsilon(y)}k(t,x,y).$$

Thus, $\hat{T}_{\epsilon,p}(t)$ and $T_{\epsilon,p}(t)$ are consistent. Choosing $\epsilon_1$ small enough, by Lemma 15.5.3 we may assume that $\lambda \in \rho(B_{\epsilon,p})$ whenever $|\epsilon| \le \epsilon_1$. Hence $\lambda \in \rho(\hat{A}_{\epsilon,p})$ whenever $|\epsilon| \le \epsilon_1$.

Since $L^p(\Omega) \subset \hat{L}^p_\epsilon$ now it follows from Corollary 15.3.5 that $R(\lambda, \hat{A}_{\epsilon,p})$ and $R(\lambda, A_{\epsilon,p})$ are consistent. Since $\hat{T}_{\epsilon,p}(t)$ and $\tilde{T}_{\epsilon,p}(t)$ have the same kernel, they are consistent. Since also $\tilde{L}^p_\epsilon \subset \hat{L}^p_\epsilon$, it follows from Corollary 15.3.5 again that $R(\lambda, \tilde{A}_{\epsilon,p})$ and $R(\lambda, \hat{A}_{p,\epsilon})$ are consistent. Hence also $R(\lambda, \tilde{A}_{\epsilon,p})$ and $R(\lambda, A_{\epsilon,p})$ are consistent. The proof of (a) is finished.

b) Finally we consider the case where $p = \infty$. Consider the adjoint semigroup $(T(t)')_{t\geq 0}$ of $(T(t))_{t\geq 0}$ which also satisfies a Gaussian estimate. Let $A$ be the generator of $(T(t))_{t\geq 0}$. The adjoint $A'$ of $A$ is the generator of $(T(t)')_{t\geq 0}$. Hence $\sigma(A') = \sigma(A)$. Denote by $(T_1'(t))_{t\geq 0}$ the extrapolation semigroup of $(T'(t))_{t\geq 0}$ on $L^1(\Omega)$ and by $A_1'$ its generator. Then $\sigma(A_1') = \sigma(A')$ by a). The generator $A_\infty$ of $(T_\infty(t))_{t\geq 0}$ is the adjoint of $A_1'$. Hence $\sigma(A_\infty) = \sigma(A_1') = \sigma(A)$. Now the proof of Theorem 15.5.1 is complete. □

## 15.6 Exercises

In the first exercise we prove an identity which occurs in the proof of Theorem 15.5.1.

**Exercise 15.6.1.** *Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup with generator $A$ and let $\lambda \in \rho(A)$. Show that*

$$R(\lambda, A) = \int_0^1 e^{-\lambda t} T(t) x dt + e^{-\lambda} T(1) R(\lambda, A) x \qquad (x \in X).$$

*(Hint: Use Proposition 2.2.6 and Exercise 2.6.2.)*

In the next exercise we consider an elliptic operator with unbounded drift. We use $L^p$-invariance of the spectrum to prove exponential stability of the semigroup in all $L^p$-spaces.

**Exercise 15.6.2.** *Assume that $\mathbb{K} = \mathbb{R}$. Let $\Omega \subset \mathbb{R}^n$ be open and contained in a strip. Let $a_{ij}$ satisfy the uniform ellipticity condition and let $c_0 : \Omega \to \mathbb{R}_+$ be measurable and $b, c \in C^1(\Omega, \mathbb{R}^n)$ such that (11.9) and (11.10) hold for some $\gamma > 0$, $0 \leq \beta < 1$. Let $H = L^2(\Omega)$, $V := \{u \in H_0^1(\Omega) : \int_\Omega c_0 |u|^2\}$.*

*a) Observe that*

$$[u|v]_V := \int_\Omega \nabla u \nabla v dx + \int_\Omega c_0 uv$$

*defines a scalar product on $V$ which induces a norm equivalent to the one considered in Section 11.4 (use Poincaré's inequality).*

*b) Consider the form $a : V \times V \to \mathbb{R}$ defined before Proposition 11.4.2. Check the proof of Proposition 11.4.2 to deduce that the form is coercive.*

*c) Consider the semigroups $(e^{-tA_1})_{t\geq 0}$ on $L^1(\Omega)$ defined in Theorem 11.4.4. Show that there exists $\epsilon > 0$, $M \geq 0$ such that*

$$\|e^{-tA_1}\|_{\mathcal{L}(L^1(\Omega))} \leq M e^{-\epsilon t} \qquad (t \geq 0).$$

*(Hint: Use the following theorem: Let $(T(t))_{t\geq 0}$ be a positive $C_0$-semigroup on an $L^1$-space with generator $A$. Then $s(A) = \omega(A)$.)*

**Remark 15.6.3.** *See e.g. [ABHN01, Prop. 5.3.7] for the proof of the theorem mentioned in the hint to the above exercise. The theorem also holds in general $L^p$-spaces. This is Weis' Theorem ( [ABHN01, Thm. 5.3.6]).*

In the following exercise we continue to investigate consistency of resolvents.

**Exercise 15.6.4** (consistency of resolvents). *Let $X, Y$ be Banach spaces such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, where $Z$ is a Banach space. Assume that $X \cap Y$ is dense in $X$ and in $Y$. Consider the Banach space $X \cap Y$ with norm*

$$\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y \qquad (u \in X \cap Y).$$

*Let $(T_X(t))_{t\geq 0}$ and $(T_Y(t))_{t\geq 0}$ be $C_0$-semigroups on $X$ and $Y$ with generators $A_X$ and $A_Y$, respectively. Assume that*

$$T_X(t)x = T_Y(t)x \qquad (x \in X \cap Y,\ t \geq 0).$$

*a) Show that there exists a unique $C_0$-semigroup $(S(t))_{t\geq 0}$ on $X \cap Y$ such that $(S(t))_{t\geq 0}$ is consistent with $(T_X(t))_{t\geq 0}$ and with $(T_Y(t))_{t\geq 0}$ $(t \geq 0)$.*

*b) Show that the generator $B$ of $(S(t))_{t\geq 0}$ is given by*

$$\begin{aligned} D(B) &:= \{x \in D(A_X) \cap D(A_Y) : A_X x = A_Y x\} \\ Bx &:= A_X x. \end{aligned}$$

*c) Show that*

$$\{\lambda \in \rho(A_X) \cap \rho(A_Y) : R(\lambda, A_X) \text{ and } R(\lambda, A_Y) \text{ are consistent }\} \subset \rho(B).$$

*d) Show that for $\lambda \in \rho(B) \cap \rho(A_X) \cap \rho(A_Y)$, $R(\lambda, A_X)$ and $R(\lambda, A_Y)$ are consistent.*

*e) Consider the semigroup defined by $T_p(t)f(x) := f(e^{-t}x)$ $(f \in L^p(0,\infty),\ x \in (0,\infty))$, where $1 \leq p < \infty$, with generator $A_p$ as in Example 15.4.1. Let $q \in (p,\infty)$. Show that there exists a unique $C_0$-semigroup $(S(t))_{t\geq 0}$ on $L^p(0,\infty) \cap L^q(0,\infty)$ consistent with $(T_p(t))_{t\geq 0}$ and $(T_q(t))_{t\geq 0}$. Denote by $B$ the generator of $(S(t))_{t\geq 0}$. Show that*

$$\sigma(B) = \left\{\lambda \in \mathbb{C} : \frac{1}{q} \leq \mathrm{Re}\lambda \leq \frac{1}{p}\right\}.$$

*f) Study a $C_0$-semigroup $(T(t))_{t\geq 0}$ on the Banach space $X + Y$ with norm*

$$\|u\|_{X+Y} := \inf\{\|x\|_X + \|y\|_Y : x \in X,\ y \in Y,\ u = x + y\}$$

*which is consistent with $(T_X(t))_{t\geq 0}$ and $(T_Y(t))_{t\geq 0}$. Establish properties that are analogous to those in a)–e).*

## 15.7 Comments

We comment on the diverse topics.

Convergence of unbounded operators is studied in detail in Kato's monography [Kat66]. A similar result to Theorem 15.5.1 is in [Kat66, p. 212]. Our notion of convergence in the resolvent sense corresponds to the convergence in the operator norm. Similarly, one may consider strong convergence for resolvents, see [ABHN01, Prop. 3.6.2] for a characterisation in terms of the generators. Upper semicontinuity of the spectrum is a classical result for bounded operators which is valid in all Banach algebras. It is usually proved by a contour integral argument.

The possible inconsistency of the resolvent of consistent semigroups (as in Example 15.4.1) was discovered in [Are94], where also the results of Sections 15.2–15.4 are taken from.

Theorem 15.5.1 has been proved in [Are94] in the case where the resolvent set of $A$ is connected. The additional argument leading to the general result is due to Kunstmann [Kun99]. The proof given here follows closely the two papers [Are94] and [Kun99]. The technique used in this proof goes back to Hempel and Voigt [HV86]– [HV87], who prove a $L^p$-spectral independence for Schrödinger operators. Kunstmann studies $L^p$-independence of spectra by kernel estimates in a systematic way in a series of papers [Kun99]– [Kun00]– [Kun01]– [Kun02]. In particular, we mention the following interesting example from [Kun02].

**Example 15.7.1.** *On an open bounded subset $\Omega \subset \mathbb{R}^2$ define in a weak sense the Neumann Laplacian $\Delta_2^\Omega$ as in Section 3.1. As we have seen in Example 11.1.7, the semigroup generated by $\Delta_2^\Omega$ is positive and irreducible; in fact, one can see that Theorem 9.3.2 applies, and the semigroup is submarkovian. By the results of Section 4.4, we can consider the family $(e^{t\Delta_p^\Omega})_{t\geq 0}$ of extrapolated semigroups on $L^p(\Omega)$, $1 \leq p \leq \infty$. Then the domain $\Omega$ can be chosen in such a way that*

$$\sigma(\Delta_p^\Omega) = S_p := \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \frac{\pi}{2} - \theta_p \right\} \cup \{0\} ,$$

*where $\theta_p \in [0, \frac{\pi}{2}]$, with $\cos\theta_p = |1 - \frac{2}{p}|$, $1 \leq p \leq \infty$. Thus, the spectrum of $\Delta_p^\Omega$ does depend on $p \in [1, \infty]$. Such a striking result is [Kun01, Thm. 4], where it is also shown that $\Omega$ can be even chosen with of measure. But of course, this $\Omega$ has not the extension property, since otherwise (by Example 11.1.7) $\Delta^\Omega$ would have compact resolvent, which contradicts Proposition 15.3.7 asserting the $L^p$-independence of the spectrum and also the fact that $\sigma(\Delta_2^\Omega)$ is uncountable..*

*Observe in particular that $\sigma(A_1)$ is the closed right halfplane. Consequently, the semigroup $(e^{t\Delta_1^\Omega})_{t\geq 0}$ is not holomorphic. By Theorem 14.2.1 it follows that $(e^{t\Delta_2^\Omega})_{t\geq 0}$ does not have a Gaussian upper bound.*

**Remark 15.7.2.** *If $-A_2$ generates a symmetric submarkovian $C_0$-semigroup on $L^2(\Omega)$, then by a result of Liskevich and Semenov [LS96] one always has $\sigma(A_p) \subset S_p$, $1 \leq p \leq \infty$. Thus, in the above example the worst case occurs.*

A different approach to spectral $L^p$-independence based on commutator estimates is given by Hieber and Schrohe [HS99]. In the self-adjoint case, Davies proves spectral $L^p$-independence under the hypothesis of Gaussian estimates by a special functional calculus due to Sjöstrand, cf. [Dav95a]– [Dav95b]. This proof is also given in Ouhabaz's monograph [Ouh04].

Spectral $L^p$-independence of the Laplace–Beltrami operator on a Riemannian manifold $M$ depends on the geometry of $M$, see Sturm [Stu93] for positive and Davies, Simon, and Taylor [DST88] for negative results in this direction.

# Epilogue

This last lecture is the end of our introduction to heat kernels. We saw how they can be used for spectral properties (Weyl's formula, $L^p$-independence of the spectrum), for regularity ($L^1$-holomorphy), and for asymptotics (irreducibility). This is only part of many interesting implications and interplays with mathematical analysis and physics.

Missing is in particular the subject of maximal regularity which opens the door to nonlinear parabolic equations. Classical approaches are based on precise knowledge of the domains of the operators. This can be obtained only for smooth coefficients and domains (Agmon–Douglis–Niremberg estimates in the $L^p$-framework, Schauder estimates for the $C^\alpha$-case). The weak formulation based on form methods, with the help of kernel estimates, gives us those analytical properties that allow us to pass to semilinear and quasilinear problems even though the coefficients might be just measurable and the domains arbitrary. In fact, maximal regularity is one more striking consequence of kernel estimates. We refer to [Are04, § 6] for a brief survey and to Denk–Hieber–Prüss [DHP03] and Kunstmann–Weis [KW04] for more detailed information.

The diverse projects of the Phase 2 of ISEM0506 will fill some of these gaps and lead to many new discoveries as will the workshop in Blaubeuren.

## The Conundrum of the Workshops

When the flush of a new-born sun fell first on Eden's green and gold,
Our father Adam sat under the Tree and scratched with a stick in the mould;
And the first rude sketch that the world had seen was joy to his mighty heart,
Till the Devil whispered behind the leaves, "It's pretty, but is it Art?"

Wherefore he called to his wife, and fled to fashion his work anew –
The first of his race who cared a fig for the first, most dread review;
And he left his lore to the use of his sons – and that was a glorious gain
When the Devil chuckled "Is it Art?" in the ear of the branded Cain.

They fought and they talked in the North and the South, they talked and they fought in the West,
Till the waters rose on the pitiful land, and the poor Red Clay had rest –
Had rest till that dank blank-canvas dawn when the dove was preened to start,
And the Devil bubbled below the keel: "It's human, but is it Art?"

They builded a tower to shiver the sky and wrench the stars apart,
Till the Devil grunted behind the bricks "It's striking, but is it Art?"
The stone was dropped at the quarry-side and the idle derrick swung,
While each man talked of the aims of Art, and each in an alien tongue.

They fought and they talked in the North and the South, they talked and they fought in the West,
Till the waters rose on the pitiful land, and the poor Red Clay had rest –
Had rest til the dank, blank-canvas dawn when the dove was preened to start,
And the Devil bubbled below the keel: "It's human, but is it Art?"

The tale is as old as the Eden Tree – and new as the new-cut tooth –
For each man knows ere his lip-thatch grows he is master of Art and Truth;
And each man hears as the twilight nears, to the beat of his dying heart,
The Devil drum on the darkened pane: "You did it, but was it Art?"

We have learned to whittle the Eden Tree to the shape of a surplice- peg
We have learned to bottle our parents twain in the yelk of an addled egg,
We know that the tail must wag the dog, for the horse is drawn by the cart;
But the Devil whoops, as he whooped of old: "It's clever, but is it Art?"

When the flicker of London sun falls faint on the Club-room's green and gold,
The sons of Adam sit them down and scratch with their pens in the mould –

They scratch with their pens in the mould of their graves, and the ink and the anguish start,
For the Devil mutters behind the leaves: “It’s pretty, but is it Art?”

Now if we could win to the Eden Tree where the Four Great Rivers, flow,
And the Wreath of Eve is red on the turf as she left it long ago,
And if we could come when the sentry slept and softly scurry through,
By the favour of God we might know as much as out father Adam knew.

Rudyard Kipling 1890

The foundations of semigroup theory were established in Einar Hille’s treatise [Hil48], which contains the first publication of Hille’s proof of the Hille–Yosida Theorem and which starts by the lines:

> And each man hears as the twilight nears,
> to the beat of his dying heart,
> The Devil drum on the darkened pane:
> “You did it, but was it Art?”

It was Eberhard Michel [Mic01] who discovered the entire poetry.

# Letters

## First Letter

Dear Participants of ISEM 2005/06, now we are ready to start!

I know that you will be disappointed: it is full of preliminaries.
There are 2 possibilities:

1. you do not have the preliminaries to understand the preliminaries: do not worry , you will get used to the new notions very soon and your friendly local coordinator will be of great help or

2. you know all this stuff. Then you can check whether you are able to do the exercises in minimal time and fill in all other proofs.

And also, it will be useful to have a common language. Maybe you share my enthusiasm for the Spectral Theorem. Selfadjoint operators are the same as multiplication operators up to unitary equivalence. This is the easiest and most useful formulation (but you can also fiddle around with spectral projections and Stieltjes integrals). Many of our examples will be selfadjoint. But the Spectral Theorem is also most useful to illustrate the form methods which will occupy a big part of the course.

The subject of ISEM2005/06 is HEAT KERNELS. There is no heat in the first lecture, but there is a kernel, the most simple one, namely a Hilbert-Schmidt kernel. We omit the proof of the well-known characterization. But I wonder whether the participants of class b) all know the criterion by which the lecture ends. It will be useful later.

At the very end of the lecture are comments. These give further information, which will not be used later, in general. This time you find Mercer's Theorem as additional information. If the kernel is continuous, one can express it by the series of eigenfunctions. Will we be able to prove continuity of heat kernels?

The excercises are compulsory. But there are others hidden in the text. Frequently they carry names like "it is easy to see", "as is well-known", "a short inspection shows"... Also the so-labelled exercises are recommended.

For the first lecture the Ulm team volunteered to put the solutions into the web. Which team will follow next week?

Now it is time to introduce the organization team: Enza Galdino is the main manager, Delio Mugnolo and Markus Biegert form the scientific committee together with the virtual lecturer.

We are hoping for interesting discussions. Critics and suggestions are most welcome.

We try to send the course each friday (10-15 pages + exerxcises and comments). Wishing you a good start,

the virtual lecturer, Wolfgang Arendt

## Second Letter

Dear Participants of ISEM2006,

there is always evolution in ISEM. But this time semigroups figure among the preliminaries (sorry). The second lecture is a concentrated presentation of semigroup theory and several results are presented without proof. The pace will lower in Lecture 3 when we will talk about more concrete things.

In today's lecture, the elementary properties of semigroups are given with proof, so that we learn how to manipulate things.

An important question is which operators generate a semigroup. We present two generation theorems. For our purposes the characterisation theorem for generators of holomorphic semigroups is the most important. We state it without proof. And in fact the contour technique used in the proof will not be used later on.

You will see that we pay particular attention to the asymptotic behaviour at 0. And indeed, we will find out later that the asymptotic behaviour of the semigroup at 0 is also responsible for the existence of a kernel and properties like ultracontractivity. It is most interesting that holomorphy of the semigroup can be described by such an asymptotic property, and for this we include the short proof. Also some of the exercises give insight to asymptotics at 0. The most beautiful is Exercise 2.6.5. You can communicate it to your friends having a beer and writing the short proof on a Bierdeckel (beer mat). This is to make you already a little familiar with German customs. More of this during the workshop in Blaubeuren June 11th to 17th, 2006 (dates for your diary!).

Till next week, best wishes

Wolfgang Arendt (virtual lecturer)

## Third Letter

Dear Participants of ISEM 06,

Lecture 3 is on the web. Preliminaries are over now with the end of lecture 2.

The main actor in the game is presented, her majesty the Laplacian.

Actually Pierre Simon Laplace (1749-1827) was made Marquis by Napoleon. He knew him since he had to take a mathematical exam with Laplace when he entered the miltary school of Saint Cyr. Napoleon did quite well as he did in his battle of Ulm , exactly 200 years ago, October 16, 1805. Ulm was Bavarian at that time, which was on the French side if I understand the story correctly . You can see the battle sight when you come to Blaubeuren, if you really want to see a battle sight.

The population of Ulm would have greatly prefered to have a visit of Laplace instead of Napoleon. We could have learnt about heat kernels 200 years before today which would have been much more delightful.

Anyway, today we learn about positivity and Poincare's inequality , which can be reformulated by saying that the heat flow decreases exponentially on strip bounded domains. The next few lectures will all be devoted to the Laplacian with Dirichlet boundary conditions. Much can be said about it, much more than the Marquis could have imagined 200 years ago.

Wishing you delightful reading,

virtually yours,

Wolfgang Arendt

# 4th Letter

Dear Participants of ISEM06,

Lecture 4 establishes the first heat kernels, governed by the Dirichlet Laplacian, our guiding example for the future.

Our favorite kernel criterion of ISEM06 is the one of Dunford-Pettis (1940), maybe first proved by L. Kantorovich and B. Vulikh: Sur la representation des operations lineaires, Comp. Math. 5 (1937) 119-165. Kantorovich wrote a book on Functional Analysis jointly with G. Akilov which contains much more material on kernel representation of operators.

It was not for his work on kernels that Kantorovich obtained the Nobel Prize in 1975, but rather for his results on the allocation of scarce sources.

Is diffusion also an economical phenomenon? We wait for further results.

For the moment being, we imagine heat diffusion. And then the comparison result Theorem 4.2.1 is plausible for physical reasons.

We work on Hilbert space. Is this natural? At least, it is easy, because of the Riesz-Frechet Lemma, as we saw last week. Still, we obtain the Lp spaces by extrapolation. We give easy direct proofs for the positive case in which we are mainly interested (see the comments for credits!). Interpolation and extrapolation is a subject which we will pursue also in the following lectures.

Virtually yours, Wolfgang Arendt (virtual lecturer)

## 5th Letter

Dear ISEM Scholars,

the fifth lecture contains a call to the public:

ABOLISH THE DICTATORSHIP OF HILBERT SPACE.

We demand continuous functions.

So our goal is to establish continuous solutions to the heat equation and also continuous kernels up to the boundary.

The lecture contains a diversion to the Dirichlet Problem, which is motivated by physical problems in electro statics and which we need to reach our goal. When was it first studied?

In 1822 Dirichlet went to Paris as a student, where he met the Marquis. But it was only later that he formulated and investigated the Dirichlet Problem (1839). One of his first results was the solution of Fermat's Theorem for n=5. We also owe to him the definition of a function, we all are used to nowadays. In fact , in 1837 he studied Fourier series and these considerations lead him to give a clear and general definition of a function. His investigations allowed him to establish the revolutionary theorem you all know:

THEOREM (Dirichlet) : FOURIER SERIES ARE DEMOCRATIC.

And indeed, Dirichlet showed that the Fourier series of a piecewise differentiable function converges pointwise to the function, where the function is continuous , and to the mean value, in points where it has a jump.

Let us come back to the Dirichlet Problem.

A complete characterisation of Dirichlet regularity in terms of barriers is due to O. Perron in 1923. And it was Norbert Wiener who characterised Dirichlet regularity by the notion of capacity shortly after. That's why one also uses the term "Wiener regular" as synonymous for Dirichlet regular.

So the main actor is Dirichlet in this lecture.

Will there be a counter revolution?

Who will be the main actor of Lecture 6? I fear, that he might be late , and exceptionally, Lecture 6 might be delayed for some days.

You have more time to devote to continuous solutions.

Virtually yours,

Wolfgang Arendt

## 6th Letter

Dear Participants of ISEM2006,

Lecture 6 is dedicated to

**RAINER NAGEL**

We all send him our very best congratulations for his 65th birthday!

He is the generator of ISEM which took place for the first time 9 years ago.

Rainer Nagel did not meet the Marquis, of course.

But the Marquis met Dirichlet in Paris as we learnt last time. Dirichlet was the successor of Gauss in Goettingen 1855. Dirichlet's successor was Riemann in 1859, and of course Riemann met Weierstrass.

Weiserstrass and Riemann had a long dispute about the Dirichlet principle which was finally settled by Hilbert (and ultimately in ISEM06).

Weierstrass was the PhD-adviser of Hermann Schwarz who obtained his PhD in 1867 , proved the Cauchy Schwarz inequality and gave a rigorous proof of the Riemann Mapping Theorem.

H. Schwarz was the PhD advisor of Lichtenstein (1909) and Lichtenstein the PhD advisor of Ernst Hoelder (1926) and Ernst Hoelder was the PhD-adviser of H.H. Schaefer and H.H. Schaefer established a Theory of Positivity in Analysis and was the PhD adviser of Rainer .

And Rainer met Jerry Goldstein and that is why the Theory of Positive Semigroups was born in 1978 (see the results in [Nag86]) .

And as you see, it is positivity which leads to the proof of Weyl's Theorem which is the subject of today's lecture. And indeed, it is positivity which leads to the existence of kernels, it is positivity which allows us to compare with the Gaussian semigroup. It is positivity which gives us the maximum principle, and positivity is the Tauberian hyposthesis in Karamata's Tauberian Theorem.

This is the story of the positivity proof of Weyl's Theorem you find in Lecture 6.

But this is only the second part of the story . Weyl's proof is different. How did he come into contact with this problem?

The story starts in 1670 when Fermat wrote his famous sentence on the margins of Diophant's arithmetica. Since then , mathematicians tried again and again to prove Fermat's Theorem (which now is Wiles' Theorem). Beginning of the 20th century, Paul Wolfskehl also failed to advance the problem . So he took over his father's industrial plant and in his testament he dedicated a prize of 100 000 Marks to the one who would find a proof of Fermat's Theorem up to September 13, 2007 (but nothing to anybody who might give a counter example). And indeed , Andrew Wiles obtained the prize in July 1997. But during many years there was no hope that the problem might be solved one day. Wolfskehl had explicitely stated in his testament that part of the money could be used by the Mathematical Institute in Goettingen to invite eminent mathematicians . It was end of October 1910 that the dutch physicist H.A. Lorentz (Lorentz Transform, Nobel Prize in Physics 1902) was invited to deliver a Wolfskehl lecture. And now we cite from Marc Kac [66]. Lorentz gave five lectures under the overall title "Alte und neue Fragen der Physik", and at the end of the fourth lecture he spoke as follows:

"In conclusion there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans. In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones on an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency intervall dv. To this end he calculates the number of overtones which lie between the frequencies v and v+dv and multiplies this number by the energy which belongs to the frequency v , and which according to a theorem of statistical mechanics is the same for all frequencies. It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lies between v and v + dv is independent of the shape of the enclosure and is simply proportional to its volume"

There is an apocryphal report that Hilbert predicted that the theorem would not be proved in his life time. Hilbert was wrong by many many years. Less than two years later, Hermann Weyl, who was present at Lorentz's lecture solved the problem and proved Theorem 3.3.5 of Lecture 6.

This is the story of Weyl's Theorem.

Virtual birthday wishes to Rainer and happy reading to everybody else,

virtually yours,

Wolfgang Arendt

## 7th Letter

Dear Participants of ISEM06,

Recall, continuous kernels were useful for the trace formula, and the trace formula could be used to estimate the eigenvalues and to prove Weyl's Theorem, and to obtain continuous kernels it was most convenient to work in spaces of continuous functions. These results were obtained for the Laplacian, to which the first part of the lectures was devoted.

We start part 2 of the lectures, which is devoted to general elliptic operators and their kernels. The continuity studies are definitely all aver now.

The methods of the following lectures are 100

The following quotation of Hilbert after listening to a talk is famous:

"Weyl, eine Sache muessen Sie mir erklaeren: Was ist das, ein hilbertscher Raum? Ich habe das nicht verstanden".

"Weyl, you should explain to me one thing: what is that exactly, a Hilbert space? I did not understand".

In fact, Hilbert had worked with quadratic forms in his 6 papers on integral equations which appeared 1904-1910. His students E. Schmidt and F. Riesz introduced spaces and operators and the final definition of Hilbert space is due to John von Neumann in 1929.

Form methods are sometimes also called "variational methods", because in the symmetric case, the solution can also be obtained by minimizing the form. And this idea goes back to Dirichlet.

The Riesz-Frechet Theorem was independently proved by F. Riesz and M. Frechet. The corresponding articles both appeared in the Comptes Rundus Acad. Sci. Paris 1907. These and more historical comments can be found in [Wer79].

In Lecture 7, the main tool is the Lax-Milgram Theorem , a decisive generalization of the Riesz-Frechet Thorem to non-symmetric forms. It appeared in P.D. Lax, A. Milgram: Parabolic equations. Contributions to the Theory of Partial Differential Equations. Annals of Math. Studies 33. Princeton 1954.

And it was this year that P. Lax obtained the Abel prize.

We are talking a lot about forms today. You might miss the kernels.

Keep in form , though,

virtually yours,

Wolfgang Arendt

Virtual Lecturer

## 8th Letter

Dear Scholars of ISEM06,

"More on Forms", this is the title of this lecture.

Still more on forms? This needs some explanations.

There are two different ways to present forms. On the one hand there is the french way, based on the notion of an elliptic form, that we chose in Lecture 7 (following J. L. Lions and the French school, see [DL88] and also [Tan79]). And then there is the anglosaxon way, as in [RS78], [Dav80], [Dav90] and [Kat66], based on the notion of a closed form. So in Lecture 8 we present closed forms and show that both notions are equivalent.

Maybe we could live with one or the other form on our way to kernels. But we do not yet know what we need on the way, and a reconsciliation might be a wise choice. In fact here in Ulm, we have suffered already centuries ago from contradictory concepts. Let us just mention the unhappy circumstances around 1700, more than 100 years before Napoleon came to visit us here in Ulm.

On November 1, 1700 , Charles II, King of Spain, died. And by his testamentary will, the Bourbon Duke Philippe of Anjou, son of the Dauphin and the bavarian Anna Maria, should be his successor. The german emperor on the other side, making valid old contracts also claimed the spanish throne. For these reasons the so called spanish succession war started. On one side France and Bavaria, and on the other the German emperor, England and Holland.

At that time Ulm was part of the German empire, actually a Freihe Reichsstadt, but very close to the bavarian border, as it still is today. In summer 1702 the small group of Ulmian soldiers were out of town, only 170 men formed the guard. This was the occasion of the bavarian assult, which happened in the following way. The bavarian lieutnant von Pechstein was sent to Ulm to explore the city for its weak points concerning defence. He stayed in the hotel am Griesbad, and discovered that one of the gates, the Gaensetor, was only protected by 13 men. So on September 8, 1702, early in the morning, several bavarian officiers, disguised as peasants in women's dresses, entered the city by the Gaensetor, overwhelmed the guard, and allowed three bavarian regiments to enter the town.

This is how the Bavarian - French occupation started. For several years bavarian and french soldiers populated the city, drinking and dancing (even in the entrance of the cathedral as the chronisist emphasises), celebrating carnival and other festivities so far unknown to the suffering population. And there was no single mathematician (as in the war of a hundred more years before, see a later lecture) to enlighten people by sciences. In June 1704, the Duke of Marlborough and Prince Eugen arrived with 30000 English and Dutch soldiers. They beat the Bavarians and French in the battle of Hochstaedt or Blenheim (close to Ulm) on September 13, 1704. (As a reward the Duke was given the beautiful palace of Blenheim close to Oxford as recognition of his big success in Ulm. Blenheim Palace is well-known as the birth place of Winston Churchill).

Anyway, it was Ulm which was the site of a big battle which finished the occupation, the drinking and dancing, but left the city in a disastrous situation.

Useless to say that people here (like everywhere) do not like battles.

For this reason, coming back to forms, we want to conduct an inquiry among the ISEM scholars.

So please fill out the following form, but not before a complete assimilation of the two forms of forms, and not after the end of ISEM 2006 (if this is possible).

| INQUIRY FROM |
|---|
| O I prefer elliptic forms |
| O I prefer closed forms |
| O No more forms, please. I want kernels |
| O In-formal comments: |
| |

Please send back the form, duly filled out, to the ISEM organisation team.

Looking forward to getting to know your opinion,

formally yours,

Wolfgang Arendt

P.S.: Historical source: D.A. Schultes: Chronik von Ulm. Ulm a.D. 1915

## 9th Letter

Dear Scholars of ISEM06,

The last lecture of 2005 is now on the website.

We describe invariance of convex sets by a beautiful criterium due to Ouhabaz. This brings us a big step further: We obtain positivity and also submarkovian semigroups operating on all Lp spaces for quite general elliptic operators.

Let us see whether the New Year brings us Kernels, Heat and Fortune.

The Ulm team wishes all of you

Merry Christmas and Happy Feasts for the End of the Year

Wolfgang Arendt
Markus Biegert
Enza Galdino
Delio Mugnolo

## 10th Letter

Dear scholars of ISEM06,

First of all let me wish you an excellent new year 2006!

In order to see how the new lecture was conceived, I have to tell you how the ISEM year 2005 ended.

This is an Internet Seminar and so we expect lot of communication. E-mails full of comments on diffusion equations, discussions about the strange ways evolution goes, chats on kernels etc. But at the end of the year, we were reminded of some young parents I met recently. Let me tell you what happened to them.

Their young boy, Paul, was already 6 years old. Still, he did not talk at all. Never had he pronounced a single word. You will understand that the parents were worried. One day, the young family was sitting around the table for dinner. Suddenly Paul opened his mouth and said loudly and clearly: "There is salt missing in the soup".

The parents, most joyful, exclaimed: "Paul, so you can speak! This is wonderful. We are so happy. But why did you not say a single word so far?" "So far, everything was allright" was Paul's answer.

In the middle of December, we were making the point on ISEM05. We were in a much better situation than Paul's parents. There were always great comments from Delft and Parma with lists of typos and shortcomings. But then arrived two mails from other places. One was a yell through the interspace: AIUTO!

The second one, from more south on the globe, was an urgent request: REDUCE THE MATERIAL, PLEASE!

Of course, we came to help right away, but it was this second email which led us to think more deeply about REDUCTION.

All this happened during these days of december, when we were preparing the pre christmas meeting of the Ulm ISEM team and friends. I was sent to this famous cook of the twin city to ask for a recipe. He looked at me with condescension, guessing that I was not able to cook a single potato. I had no choice but to reveal my identity as virtual lecturer. This changed his attitude completely and, henceforth, he treated me as a colleague. He knew, there is no cooking without heat kernels. He took much care to explain me that the most important action in cooking is to REDUCE the sauce. Then he designed a tricky and complicated algorithm to do so.

My task was more of a theoretical kind, and I did transmit the algorithm and knowledge to more competent hands. That is why it became so delicious .

And that is why the remarkable result gave the inspiration for Lecture 10: IRREDUCIBILITY

is the title. Of course, it is just an extract. There are exercises and comments, and 10 pages of text , which is IRREDUCIBLE though, you will understand.

I hope you like the sauce.

Wolfgang Arendt Virtual Catering

## 11th Letter

There was no letter !

## 12th Letter

Dear Scholars of ISEM06,

even though you had the delicacy not to write complaining letters, I know what you are thinking:

What is the use of the best sauce, as delicious and irreducible it might be, if no KERNELS are on the table?

They are back, the kernels, in Lecture 12. And it is surprising, as bad as the coefficients might be, we always produce a measurable kernel, irreducible ones, of course.

The method is most simple: it is a pure semigroup property which allows us to extrapolate things to infinity (Section 1 and 2).

Nash's inequality is offered as a desert. You will be able to digest the next Lecture 13 without it, if you want.

But who wants to leave the table without desert?

Bon Appetit.

Wolfgang Arendt ISEM-Internet Cafe

# 13th Letter

Dear Scholars of ISEM2006,

GAUSSIAN ESTIMATES form the subject of Lecture 13. Finally Gauss enters the scene, the prince of mathematics (as E.T. Bell calls him in "Men of Mathematics"). Much has been written about him, who was discovered as a Wunderkind at early age. In his thesis in Helmstedt 1799 he gave the first proof of the Fundamental Theorem of Algebra. We started talking about Laplace, le Marquis, we heard about Dirichlet, who met Laplace in Paris as a young student and who became the successor of Gauss in Goettingen. Gauss did not meet Laplace. Born in Braunschweig (north of Goettingen) he staid his entire life in the region. Soon after his extraordinary talents were discovered, he found in the Duke Ferdinand a sponsor, who gave him a modest but sufficient material basis for himself and his family which allowed him to dedicate his life to mathematics. This basis was suddenly destroyed when Duke Ferdinand was put in command of the Prussian forces. He was desastreously defeated and mortally wounded during the battle of Jena against Napolean in 1806. Fortunately, Gauss obtained a position at the observatory of Goettingen. Still, with three children, after the death of his first wife, he was in a difficult situation. The war made his situation worse. In fact, in order to govern Germany according to their ideas the victors of Jena fined the losers for more the traffic would bear. As professor and astromer at Goettingen, Gauss was rated to be good for an involuntary contribution of 2000 francs to the Napolean war chest. This exorbitant sum was quite beyond Gauss' ability to pay, even though he lived a most simple life. In this situation , Gauss received a friendly little note from Laplace, telling him that the famous French mathematician had paid the 2000 franc fine for the greatest mathematition in the world and that he considered it an honour to be able to lift this unmerited burdon from his friend's shoulders.

Gauss had 7 PhD students. The following is a folk theorem.

Theorem . Almost every German mathematician is a descendant of Gauss.

Example. In the following list the predecessor is the PhD advisor of the successor.

| | |
|---|---|
| Gauss | Helmstedt 1799 |
| Bessel | Göttingen 1810 |
| Scherk | Berlin 1823 |
| Ernst Kummer | Halle 1831 |
| Hermann Schwarz | Berlin 1864 |
| Leon Lichtenstein | Berlin 1909 |
| Ernst Hoelder | Leipzig 1926 |
| H.H. Schaefer | Leipzig 1951 |
| W. Arendt | Tübingen 1979 |
| El Maati Ouhabaz | Besancon 1992 |
| Cesar Poupaud | Bordeaux 2005 |

Exercise. Give a proof of the Theorem Hint: Gauss has 29792 descendants.

Given a mathematician, in view of the Theorem, it is not so much the problem to prove the existence of a link to Gauss. The mathematical problem consists rather in giving estimates of the number of scientific generations leading to Gauss.

Those are the so called GAUSSIAN ESTIMATES.

Virtually yours, Wolfgang Arendt

## 14th Letter

Dear Scholars of ISEM2006,

close to the end of the first phase of ISEM2006, let us go back to the foundations of modern mathematics, to the bases which allow us our studies of heat kernels and their effects on regularity and asymptotics as in today's Lecture 14.

The starting point of modern mathematics falls in war time. It is the Thirty Years' War which ravages Europe. But this time it brings not Napoleon, not the Bavarian-French occupiers and English liberators to Ulm, as 200 and 100 years before, but it brings us - a mathematician.

We cite the first lines of a book by Philip J. Davis and Reuben Hersh:

The modern world, our world of triumphant rationality, was born on November10, 1619, with a revelation and a nightmare. On that day, in a room in the small Bavarian village of Ulm, Rene Descartes, a Frenchman, twenty-three years old, crawled into a wall stove and, when he was well warmed, had a vision. It was the vision of the unification of all science.

And E.T. Bell calls November10, 1619, the official birthday of analytical geometry and of modern mathematics. He tells Descartes' dreams as follows:

In the first dream Descartes was revolved by a whirlwind and terrified by phantoms. In his second dream he found himself observing a terrific storm with the unsuperstitious eyes of science, and he noted that the storm, once seen for what it was, could do him no harm. In the third dream, all was quiet and contemplative. An anthology of poetry lay on the table . He opened it at random and read the verse of Ausonius "Quod vitae sectabor iter" (What path shall I take in life?).

Descartes said that he was filled with enthusiasm and that there had revealed to him, as in the second dream, the magic key which would unlock the treasure house of nature and put him in possession of the true foundation. What is this magic key? It is the exploration ot natural phenomena by mathematics.

What was Descartes doing in Ulm? After his studies at the Jesuit College La Fleche in Paris, at the age of eighteen, he found life in Paris too disturbing. To get a little peace, Descartes decided to go to war. First he went to Holland under Prince Maurice of Orange. Then he enlisted under the Elector of Bavaria, then waging war against Bohemia. In the winter quarters near Ulm he found tranquillity and repose.

We do not know exactly what he discovered in Ulm. Of course he is the inventor of Analytical Geometry , but he also discovered virtual velocity in mechanics. Had he also a vision of virtual lecturing on heat kernels?

We still try to find the wall stove.

Virtually yours, Wolfgang Arendt

P.S. The citations are from:

Philip J. Davis, Reuben Hersh: Descartes' Dream. The world according to mathematics. Pinguin Books . London 1986

E.T. Bells: Men of Mathematics.New York 1932

## 15th Letter

Dear Scholars of ISEM05/06,

the last lecture is devoted to Spectral Theory.

The mathematical spectrum, as we use it today, had been defined by David Hilbert in his work on integral equations early last century. Was it by chance or by ingeneous intuition, that he chose this denomination? Only 25 years later the meaning could be fully understood. In fact, it was von Neumann who introduced the notion of unbounded self-adjoint operators and who gave the mathematical formulation of quantum theory in his book: "Mathematische Grundlagen der Quantentheorie" from 1932. And this formulation is still entirely valid today with greatest success. Thus, an observable is described by a self-adjoint operator, and the point spectrum corresponds to pure states. For example, the hydrogen atom is described by a Schroedinger operator, i.e. the Laplacian plus the Coulomb potential of the electron, and the mathematical spectrum of this operator is precisely the spectrum of the atom we see.

In today's lecture , we learn that the spectrum does not depend on the functional space we choose. This contrasts these equations coming from mathematical finance. Indeed, Example 15.4.3 is nothing else than the Black-Scholes Equation.

So let us return to the Heat Equation . Our lectures started with Laplace. What was his motivation to study this subject?

Laplace lived from March 23, 1827 to March 5, 1827. During his studies of Theology at the University of Caen, he discovered his love of mathematics and his teacher Laplace's great mathematical talents. Laplace did not finish his studies at Caen but went to Paris where he studied under direction of d'Alembert. His early work were major cotributions to differential equations, to mathematical astronomy and to the theory of probability, subjects on which he worked throughout his life.

It was in 1780 when Laplace made an excursion into a new area of science. Applying quantative methods to a comparison of living and nonliving systems, Laplace and the chemist Antoine Lavoisier, with the aid of an ice calorimeter that they had invented, showed respiration to be a form of combustion. Athough Laplace soon returned to his study of mathematical astronomy, this work with Lavoisier marked the beginning of a third important area of research for Laplace, namely his work in physics, particularly on the theory of heat which he worked on towards the end of his career.

Laplace lived in a period full of political turbulence. During the Reign of Terror in 1893, together with Lavoisier, he was thrown off the commission to standardise measures, becasuse of his lack of "Republican virtues and hatred of kings". He left Paris and lived 50km southeast of Paris until July 1794. The founder of modern Chemistry, Antoine Lavoisier, was guillotined in May 1794.

You remember that Laplace examined and passed the 16 year old Napoleon in 1785. Napoleon had great admiration for mathematics and under his reign, Laplace became Minister of the Interior in 1799. Napoleon explained in his memoirs why he removed Laplace from office merely after 6 weeks: "...because he brought the spirit of the infinitely small into the government".

You know that Laplace never came to Ulm, but Napoleon did, not exactly for scientific reasons. But, for scientific reasons, he founded the Ecole Normale in Paris in 1795 and Laplace was one of the first to teach there. L'Ecole Normale is situated in the Rue d'Ulm, and that is one reason why, today, Ulm is rather connected with mathematics than with battles.

Wishing you an enjoyable and successful phase 2 of ISEM05/06,
virtually yours,
Wolfgang Arendt

# Bibliography


[AB94] W. Arendt, A.V. Bukhvalov, *Integral representations of resolvents and semigroup*, Forum Math. **6** (1994), 111–135.

[ABHN01] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics **96**, Birkhäuser, Basel 2001.

[Ada75] R. Adams, *Sobolev Spaces*, Pure and applied mathematics **65**, Academic Press, New York - San Francisco - London 1975.

[ArBe99] W. Arendt, Ph. Bénilan, *Wiener regularity and heat semigroups on spaces of continuous functions*, Topics in Nonlinear Analysis, (eds. J. Escher, G. Simonett), Birkhäuser, Basel 1998, pp. 29–49.

[AGG06] W. Arendt, G. Goldstein, J. Goldstein, *On Outgrowth of Hardy Inequality*, Preprint.

[AMP06] W. Arendt, G. Metafune, D. Pallara, *Schroedinger operators with unbounded drift*, J. Operator Th., **55** (2006) 185–211.

[AN00] W. Arendt and N. Nikolski, *Vector-valued holomorphic functions revisited*, Math. Z. **234** (2000), 777–805.

[AN06] W. Arendt and N. Nikolski, *Addendum: Vector-valued holomorphic functions revisited*, Math. Z. **252** (2006), 687–689.

[AtE97] W. Arendt and A. F. M. Ter Elst, *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Operator Theory **38** (1997), 87–130.

[Are94] W. Arendt: *Gaussian estimates and interpolation of the spectrum in $L^p$*. Diff. Int. Equ. **7** (1994), 1153–1168.

[Are97] W. Arendt, *Semigroup properties by Gaussian estimates*, Nonlinear Evolution Equations and Applications, RIMS, Kyoto, Kokyuroku **1009** (1997), 162–180.

[AHLMT02] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, Ph. Tchamitchian: *The solution of the Kato square root problems for second order elliptic operators on $R^n$*. Ann. Math. **156** (2002), 633–654.

[Are00] W Arendt, *Resolvent positive operators and inhomogeneous boundary conditions*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. (4) **29** (2000) 639–670.

[Are04] W. Arendt, *Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates*, HANDBOOK OF DIFFERENTIAL EQUATIONS, Evolutionary Equations, Vol. 1, C.M. Dafermos, E. Feireisl eds., Elsevier 2004, 1–85.

[Aro68] D. G. Aronson. *Nonnegative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa **22 (3)** (1968) 607–694.

[Bie00] M. Biegert, *Sobolev Räume*, Master Thesis, Universität Ulm, 2000.

[BH91] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space*, de Gruyter Studies in Mathematics **14**, Walter de Gruyter, Berlin, 1991.

[Bre83] H. Brézis, *Analyse Fonctionelle*, Masson, Paris 1983.

[Buc94] Z. Buczolich, *Product sets in the plane, sets of the form $A + B$ on the real line and Hausdorff measure*, Acta Math. Hungar. **65** (1994) 107–113.

[BW02] M. Biegert, M. Warma, *Sobolev functions whose weak trace at the boundary is zero*, Ulmer Seminare Heft **7** (2002), 109–121.

[Cou90] T. Coulhon, *Dimension á l'infini d'un smi-groupe analytique*, Bull. Sci. Math. **114** (1990), 485–500.

[CH93] R. Courant, D. Hilbert, *Methoden der Mathematischen Physik*, Springer, Berlin 1993.

[Con78] J.B. Conway, *Functions of One Complex Variable*, Springer, Berlin 1978.

[Dan00] D. Daners, *Heat kernel estimates for operators with boundary conditions*, Math. Nachr. **217** (2000), 13–41.

[Dan05] D. Daners, *Dirichlet problems on varying domains*, J. Diff. Eqns. **188** (2003) 591–624.

[DST88] E.B. Davies, B. Simon, and M. Taylor, *$L^p$ spectral theory of Kleinian groups*, J. Funct. Anal. **78** (1988), 116–136.

[Dav80] E.B. Davies, *One-parameter Semigroups*, Academic Press, London 1980.

[Dav87] E.B. Davies, *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math. **109** (1987), 319–333.

[Dav89] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics **92**, Cambridge University Press, Cambridge 1989.

[Dav95a] E.B. Davies, *$L^p$ spectral independence and $L^1$ analyticity*, J. Lond. Math. Soc. **52** (1995), 177–184.

[Dav95b] E.B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics **42**, Cambridge Univ. Press, Cambridge, 1995.

[DL88] R. Dautray, J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vols 1–3, Springer, Berlin 1988.

[DHP03] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Am. Math. Soc. **788**, 2003.

[Dod81] J. Dodziuk, *Eigenvalues of the Laplacian and the heat equation.*, Amer. Math. Monthly **88** (1981) 686–695.

[DvC00] M. Demuth, J. van Casteren, *Stochastic Spectral Theory for Selfadjoint Feller Operators*. Birkhäuser, Basel 2000.

[EE87] D.E. Edmunds, W.D. Evans, *Spectral Theory and Differential Operators*, Oxford 1987.

[EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics **194**, Springer-Verlag, Berlin 2000.

[Eva98] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence, RI 1998.

[FS86] E.B. Fabes and D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational. Mech. Anal. **96** (1986), 327–338.

[FOT94] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics **19**, Walter de Gruyter, Berlin, 1994.

[Gol85] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, Oxford 1985.

[GT98] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin 1998.

[Haa04] M. Haase, *Convexity inequalities for positive operators*, Ulmer Seminare, Heft **9** (2004), 212–222.

[HV86] R. Hempel and J. Voigt, *The spectrum of a Schrdinger operator in $L_p(\mathbf{R}^\nu)$ is p-independent*, Commun. Math. Phys. **104** (1986), 243–250.

[HV87] R. Hempel and J. Voigt, *On the $L_p$-spectrum of Schrödinger operators*, J. Math. Anal. Appl. **121** (1997), 138–159.

[HS99] M. Hieber and E. Schrohe, *$L^p$ spectral independence of elliptic operators via commutator estimates* Positivity **3**, 259–272.

[Hie96] M. Hieber, *Gaussian estimates and holomorphy of semigroups on $L^p$ spaces*, J. London Math. Soc. **54** (1996), 148–160.

[Hil48] E. Hille, *Functional analysis and semi-groups*, AMS Colloquium Publications **31**, AMS, New York, 1948.

[Hoc73] H. Hochstadt, *Integral Equations*, Wiley, New York 1973.

[ISEM99/00] W. Arendt, *Semigroups Generated by Elliptic Operators*, `www.mathematik.uni-ulm.de/m5/arendt/isemskript.dvi`.

[Kac66] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly, **73** (1966), no. 4, part II, 1–23.

[Kar31] J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplace'sche und Stieljes Transformation betreffen*, J. Reine Angew. Math., **164** (1931), 27–39.

[Kat66] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1966.

[Kun99] P.C. Kunstmann, *Heat kernel estimates and $L^p$ spectral independence of elliptic operators*, Bull. Lond. Math. Soc. **31** (1999), 345–353.

[Kun00] P.C. Kunstmann, *Kernel estimates and $L^p$-spectral independence of differential and integral operators*, in "Operator theoretical methods" (Proceedings Timisoara 1998) (eds. A. Gheondea et al.), The Theta Foundation, Bucarest, 2000, 197–211.

[Kun01] P.C. Kunstmann, *Uniformly elliptic operators with maximal $L^p$-spectrum in planar domains*, Arch. Math. **76** (2001), 377–384.

[Kun02] P.C. Kunstmann, *$L^p$-spectral properties of the Neumann Laplacian on horns, comets and stars*, Math. Z. **242** (2002), 183–201.

[KW04] P.C. Kunstmann and L. Weis, *Maximal $L_p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus*, in: "Functional analytic methods for evolution equations" (Proceedings Levico Terme 2001) (eds. M. Iannelli et al.), Lecture Notes in Mathematics **1855**, Springer Verlag, Berlin, 2004, 65–311.

[LS96] V.A. Liskevich and Yu.A. Semenov, *Some problems on Markov semigroups*, in "Schrdinger operators, Markov semigroups, wavelet analysis, operator algebras" (eds. M. Demuth et al.), Math. Top. **11**, Akademie Verlag, Berlin, 1996, 163–217.

[LP76] G. Lumer, L Paquet, *Semi-groupes holomorphes, produit tensoriel de semi-groupes et équations d'évolution*, Séminaire: Théorie du Potentiel, Lecture Notes in Math. Vol. 563, Springer, 1976, pp. 202–218.

[Lun95] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser, Basel 1995.

[MaRö92] Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992.

[MR83] A. Majekowski, BD.W. Robinson, *Strictly positive and strongly positive semigroups*, J. Austr. Math. Soc. Ser. A **34** (1983), 36–48.

[McI82] A. McIntosh, *On representing closed accretive sesquilinear forms as* $(A^{1/2}u, A^{1/2}v)$*.* Collège de France Seminar Vol. III, H. Brézis and J. L. Lions Eds., Pitman RNM **70** (1982), 252–267.

[Mer09] T. Mercer, *Functions of positive and negative type, and their connection with the theory of integral equations*, Phil. Trans. Roy. Soc. London (A), **209** (1909), 415–446.

[MN91] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1999.

[Mic01] E. Michel, *Zur Spektraltheorie elliptischer Differentialoperatoren*, Diploma Dissertation, Tübingen 2001.

[Nag86] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Math. **1184**, Springer-Verlag, Berlin 1986.

[Nec67] J. Nečas, *Les Méthodes Directes en Théorie des Equations Elliptiques*, Masson, Paris 1967.

[Ouh92a] E.-M. Ouhabaz, *Propriétés d'Ordre et de Contractivité pour les Semi-Groupes et Applications aux Opérateurs Elliptiques*, Ph.D. Thesis, Université de Franche-Compté, Besancon 1992.

[Ouh92b] E.-M. Ouhabaz, *$L^\infty$-contractivity of semigroups generated by sectorial forms*, J. Lond. Math. Soc., II. Ser. **46** (1992), 529-542.

[Ouh96] E.M. Ouhabaz, *Invariance of closed convex sets and domination criteria for semigroups*, Potential Anal. **5** (1996), 611-625.

[Ouh04] E.M. Ouhabaz, *Gaussian upper bounds for heat kernels of second-order elliptic operators with complex coefficients on arbitrary domains*, J. Operator Th. **51** (2004), 335–360.

[Ouh05] E.M. Ouhabaz, *Analysis of Heat Equations on Domains*, LMS Monograph Series **31**, Princeton University Press, Princeton 2005.

[dPa86] B. de Pagter, *Irreducible compact operators*, Math. Z. **192** (1986), 149–153.

[Paz83] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York 1983.

[RS72] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol. 1*, Academic Press, New York - San Francisco - London 1972.

[RS78] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol. 4*, Academic Press, New York 1978.

[Rob91] D.W. Robinson: *Elliptic Operators on Lie Groups*, Oxford University Press, Oxford, 1991.

[Rud91] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw–Hill, New York 1991.

[Sal02] L. Saloff-Coste, *Aspects of Sobolev-type Inequalities*, Cambridge University Press, Cambridge, 2002.

[Sch74] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin 1974.

[Sim79] B. Simon, *Functional Integration and Quantum Physics*, Acedemic Press, London 1979.

[SB02] Z. Sobol, H. Vogt, *On the $L^p$-theory of $C_0$-semigroups associated with second order elliptic operators*, J. Funct. Anal. **193** (2002), no. 1, 24–54.

[Ste70] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press 1970.

[Stu93] K.T. Sturm, *On the $L^p$-spectrum of uniformly elliptic operators on Riemannian manifolds*, J. Funct. Anal. **118** (1993), 442–453.

[Tan79] H. Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics. 6. London - San Francisco - Melbourne: Pitman (1979).

[VSC] N.T. Varopoulos, L. Saloff-Coste, T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics 100. Cambridge University Press, Cambridge 1992.

[VSC93] N. Varopoulos, L. Saloff-Coste, T. Coulhon, *Geometry and Analysis on Groups*, Cambridge University Press, 1993.

[vNe55] J. von Neumann, *Mathematical Foundations of the Quantum Mechanics*, Princeton University Press, Princeton 1955.

[Vog01] H. Vogt, *$L^p$-properties of second order elliptic differential operators*, Dissertation 2001.

[Voi86] J. Voigt, *Absorption semigroups, their generators, and Schrdinger semigroups*, J. Funct. Anal. **67**, 167-205 (1986).

[Voi92] J. Voigt, *One-parameter semigroups acting simultaneously on different $L_p$-spaces*, Bull. Soc. Roy. Sci. Liège **61** (1992), 465–470.

[Wer97] D. Werner, *Funktionalanalysis*, Springer, Berlin 1997.

[Wey11] H. Weyl, *Über die asymptotische Verteilung der Eigenwerte*, Gött. Nachr., 1911, 110–117.

[Wey12] H. Weyl, *Das asymptotische Verhalten der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), 441–479.

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